Homework Assignment 6 — Solutions

- Q8.1
  Room temperature is 294 K, so \( kT = 4.06 \times 10^{-14} \text{ J} = 0.0253 \text{ eV} \approx 1/40 \text{ eV} \). \( kT = 1 \text{ eV} \) when \( T = 11,600 \text{ K} \), and \( kT = 13.6 \text{ eV} \) when \( T = 158,000 \text{ K} \).

- Q8.5
  The Boltzmann equation gives us
  \[ \frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-(E_2-E_1)/kT} = 4 e^{-10.2 \text{ eV}/kT}, \]
  so we have \( \frac{n_2}{n_1} = 1\% \) when \( e^{-10.2 \text{ eV}/kT} = 0.0025 \). Taking the logarithm of both sides gives \( -10.2 \text{ eV}/kT = -5.99 \), and solving for \( T \) gives \( T = 1.97 \times 10^4 \text{ K} \). The same procedure for \( \frac{n_2}{n_1} = 10\% \) gives \( T = 3.21 \times 10^4 \text{ K} \).

- Q8.6
  \( a \).
  From the Boltzmann equation,
  \[ \frac{n_3}{n_1} = \frac{g_3}{g_1} e^{-(E_3-E_1)/kT} = 9 e^{-12.1 \text{ eV}/kT}, \]
  which gives \( \frac{n_3}{n_1} = 1 \) when \( T = 6.38 \times 10^4 \text{ K} \).
  
  \( b \).
  When \( T = 85,400 \text{ K} \), \( \frac{n_3}{n_1} = 9 e^{-1.64} = 1.74 \), so if there are \( N_1 = N \) atoms in the \( n = 1 \) state then there are \( N_3 = 1.74 N \) atoms in the second excited state \( (n = 3) \).

  \( c \).
  As \( T \to \infty \), the exponential factor in the Boltzmann equation becomes 1 for every value of \( n \), so the Boltzmann equation predicts that the distribution of electrons mirrors the values of the degeneracies: level \( n \) has a number of electrons proportional to \( n^2 \).
  In reality, as \( T \to \infty \) all atoms ionize, and all the electrons are free electrons.

- Q8.7
  We have \( Z_I = g_1 + g_2 e^{-10.2 \text{ eV}/kT} + g_3 e^{-12.1 \text{ eV}/kT} \), with \( g_1 = 2(1)^2 = 2 \), \( g_2 = 2(2)^2 = 8 \), and \( g_3 = 2(3)^2 = 18 \). At \( T = 10,000 \text{ K} \), this gives \( Z_I = 2 + 5.81 \times 10^{-5} + 1.46 \times 10^{-5} \). We can see that the second and third terms are much smaller than \( g_1 = 2 \), so we have \( Z_I \approx 2 \).

- Q8.9
  \( a \).
  The Saha equation is
  \[ \frac{N_{II}}{N_1} = \frac{2Z_{II}}{n_e Z_I} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_I/kT}. \]
  We also have charge conservation
  \[ n_e V = N_{II} \]
  and conservation of nucleons
  \[ N_i = \rho V / (m_p + m_e) \approx \rho V / m_p, \]
  where \( N_i = N_1 + N_{II} \) is the total number of hydrogen nuclei (whether in neutral hydrogen or ionized hydrogen).
  The first step is to replace the partition function \( Z_I \) with the ground state degeneracy \( g_1 = 2 \)— recall from question 8.7 that this is a good approximation at 10,000 K, and in fact it is a good
approximation for $kT \ll 10.2 \text{ eV}$, or $T \ll 120,000 \text{ K}$. The partition function $Z_{\text{II}}$ of the proton is 1. We can now write the Saha equation (for this pure hydrogen case, and assuming $T \ll 120,000 \text{ K}$) as

$$\frac{N_{\text{II}}}{N_{\text{I}}} = \frac{1}{n_e} \left(\frac{2\pi m_e kT}{\hbar^2}\right)^{3/2} e^{-\chi_1/kT} = \frac{V}{N_{\text{II}}} \left(\frac{2\pi m_e kT}{\hbar^2}\right)^{3/2} e^{-\chi_1/kT},$$

where we use charge conservation to get rid of $n_e$ in the final expression. We also have $N_{\text{I}} = N_{\text{I}} + N_{\text{II}} = N_{\text{I}} (1 + N_{\text{II}}/N_{\text{I}})$; from the Saha equation, we get $N_{\text{I}} = N_{\text{I}} (1 + x/N_{\text{II}})$, where $x = V \left(\frac{2\pi m_e kT}{\hbar^2}\right)^{3/2} e^{-\chi_1/kT}$. We can rearrange this as $N_{\text{I}} = N_{\text{I}} / (1 + x/N_{\text{II}})$. Putting this in the Saha equation gives

$$\frac{N_{\text{II}}}{N_{\text{I}} (1 + x/N_{\text{II}})} = \frac{x}{N_{\text{II}}},$$

or

$$\frac{N_{\text{II}}^2}{N_{\text{I}}} (1 + x/N_{\text{II}}) = x.$$

This gives

$$\frac{N_{\text{II}}^2}{N_{\text{I}}} + x \frac{N_{\text{II}}}{N_{\text{I}}} - x = 0.$$

Dividing through by $N_{\text{I}}$, we have

$$\left(\frac{N_{\text{II}}}{N_{\text{I}}}\right)^2 + \frac{x}{N_{\text{I}}} \left(\frac{N_{\text{II}}}{N_{\text{I}}}\right) = \frac{x}{N_{\text{I}}} = 0. \quad (1)$$

Now we use the conservation of nucleons, $N_{\text{I}} \approx \rho V/m_p$ and our expression for $x$, $x = V \left(\frac{2\pi m_e kT}{\hbar^2}\right)^{3/2} e^{-\chi_1/kT}$, to get

$$\frac{x}{N_{\text{I}}} = \frac{V \left(\frac{2\pi m_e kT}{\hbar^2}\right)^{3/2} e^{-\chi_1/kT}}{\rho V/m_p} = \left(\frac{m_p}{\rho}\right) \left(\frac{2\pi m_e kT}{\hbar^2}\right)^{3/2} e^{-\chi_1/kT},$$

which we plug in to equation 1 to get

$$\left(\frac{N_{\text{II}}}{N_{\text{I}}}\right)^2 + \left(\frac{m_p}{\rho}\right) \left(\frac{2\pi m_e kT}{\hbar^2}\right)^{3/2} e^{-\chi_1/kT} - \left(\frac{m_p}{\rho}\right) \left(\frac{2\pi m_e kT}{\hbar^2}\right)^{3/2} e^{-\chi_1/kT} = 0.$$

(b) Taking $\rho = 10^{-6} \text{ kg m}^{-3}$, we have

$$\left(\frac{N_{\text{II}}}{N_{\text{I}}}\right)^2 + 4.04 (T/1 \text{ K})^{3/2} e^{-158,000 \text{ K}/T} \left(\frac{N_{\text{II}}}{N_{\text{I}}}\right) - 4.04 (T/1 \text{ K})^{3/2} e^{-158,000 \text{ K}/T} = 0.$$

Defining $b \equiv 4.04 (T/1 \text{ K})^{3/2} e^{-158,000 \text{ K}/T}$, we have the quadratic equation $y^2 + by - b = 0$, where $y$ is the ionization fraction $N_{\text{II}}/N_{\text{I}}$. When $b \gg 1$, the quadratic term becomes negligible (remember, $y$ must be between 0 and 1, so $y^2$ is also between 0 and 1), and the equation reduces to $by - b \approx 0$, which has solution $y \approx 1$. When $b \ll 1$, the linear and constant terms become negligible and we have $y^2 \approx 0$, so $y \approx 0$. Plugging in $T = 5000 \text{ K}$ gives $b = 2.7 \times 10^{-8} \ll 1$, and plugging in $T = 25,000 \text{ K}$ gives $b = 2.9 \times 10^4 \gg 1$, so we expect that our graph will go from $y = N_{\text{II}}/N_{\text{I}} \approx 0$ at the low temperature end to $N_{\text{II}}/N_{\text{I}} \approx 1$ at the high temperature end, with a transition at temperatures where $b \approx 1$ (in fact, $N_{\text{II}}/N_{\text{I}} = 0.5$ when $b = 0.5$). Looking at the plot, we see that these predictions are correct—the transition from fully neutral to fully ionized happens at a temperature of about 9900 K, at which point $b \approx 0.5$. 

2
Q8.13
From Example 8.1.5, we have $T = 5777$ K and $P_e = 1.5\, \text{N m}^{-2}$ in the solar photosphere, and $Z_{\text{II}} = 2.30$ for calcium. We are also given $\chi_{\text{II}} = 11.9\, \text{eV}$ and $Z_{\text{III}} = 1$. With this, we can write the Saha equation for the ratio of doubly-ionized calcium to singly-ionized calcium:

$$\frac{N_{\text{III}}}{N_{\text{II}}} = \frac{2kT Z_{\text{III}}}{P_e Z_{\text{II}}} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_{\text{II}}/kT} = 0.002.$$ 

Very little calcium is doubly ionized in the Sun’s photosphere; instead, the calcium is almost all in the singly-ionized state—Example 8.1.5 shows that the ratio of singly-ionized calcium to neutral calcium is very high, and that almost all of the singly-ionized calcium is in the ground state. This means that almost all of the calcium in the Sun’s photosphere can contribute to the formation of the H and K lines.

Q8.14
The Saha equation is

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_i/kT}.$$ 

We can see that the ratio $N_{i+1}/N_i$ increases as the temperature $T$ increases, and that it decreases as the electron density $n_e$ increases. If we consider a main-sequence star and a giant star of the same spectral type, the fact that they have the same spectral type means that they should have (nearly) the same ratio $N_{i+1}/N_i$ for all species so that they form the spectral lines with the same relative strengths. The fact that the giant star has a lower atmospheric density means that it has a lower value of $n_e$ than the main-sequence star, so in order to maintain the same value of $N_{i+1}/N_i$ as the main-sequence star the giant star must also have a lower value of $T$ than the main-sequence star.

Q8.16
Fomalhaut has $V = 1.19$; from the H-R diagram, it also has $M_V \approx 2$. We can solve for the distance to Fomalhaut using $V - M_V = 5 \log_{10} \left( \frac{d}{10\, \text{pc}} \right)$, which gives $d \approx 7\, \text{pc}$.
Hydrogen ionization fraction vs. temperature

$\frac{N_{\text{ii}}}{N_{\text{H}}}$

Temperature (K)