1 Weighing a galaxy cluster

Suppose an X-ray telescope observed a galaxy cluster. The on-board spectrometer detects a spectrum that is consistent with thermal Bremsstrahlung at a temperature of $kT = 5\,\text{keV}$. The total photon count rate determined from the spectral fit in the energy band from $0.5\,\text{keV}$ to $2\,\text{keV}$ is

$$\Phi_{0.5-2\,\text{keV}} = 2 \times 10^{-3}\,\text{cm}^{-2}\,\text{s}^{-1}$$  \hfill (1.1)

This emission is smoothly varying over the image and is coming from within a circular region with angular diameter of

$$\Theta = 3.4\,\text{arcminutes}$$  \hfill (1.2)

The cluster is at a distance of $D = 1\,\text{Gpc} = 10^9\,\text{pc}$, where $1\,\text{pc} = 3.08 \times 10^{18}\,\text{cm}$.

**A)** What is the total photon rate emitted by the cluster (i.e., photons per second) in the $0.5\,\text{keV}$ to $2\,\text{keV}$ band?

**B)** From the notes, derive an expression for the photon emissivity (photons emitted per volume per time per frequency interval) for thermal Bremsstrahlung, $\phi_\nu$, in units of $\text{s}^{-1}\text{cm}^{-3}\text{Hz}^{-1}$.

**C)** Using the measured temperature of $5\,\text{keV}$, integrate $\phi_\nu$ from part B over frequency in the limits from $0.5\,\text{keV}$ to $2\,\text{keV}$. Do this by writing the exponential as a power series, then integrate the resulting sum. To evaluate the integral, truncate the sum when its value changes by less than 1% by adding another term (i.e., when the approximation reaches an accuracy of order 1%). Hint: This should not take more than a handful of terms.

**D)** Assume the cluster is composed of only hydrogen and helium, both fully ionized and with a relative mass ratio of 3 to 1 (roughly solar abundances). Determine the number density of electrons $n_e$ in terms of the number density of hydrogen nuclei, $n_H$. Determine the effective $Z^2$ for the ions in the plasma in terms of $n_H$ (this is a weighted average of $Z^2$). Re-write your expression from part C in terms of $n_H$ only. You may assume $\bar{g}_\text{H} = 1$.

**E)** Assume the cluster is a uniform gaseous sphere of the angular diameter measured in the X-rays (i.e., uniform density inside, zero density outside). Defining the hydrogen emission measure

$$\text{EM}_H \equiv \int_V dV \, n_H^2$$  \hfill (1.3)

determine $\text{EM}_H$ for the observed count rate of the cluster and your formula for the photon emissivity from part D.
F) Determine the total gas mass of the cluster. Express your answer in units of solar masses.

G) Assuming the plasma behaves like an ideal monatomic gas, determine the total thermal energy $E_{\text{th}}$ of the cluster.

H) Determine the total integrated Bremsstrahlung luminosity $L_X$ of the cluster.

I) Determine the instantaneous radiative cooling time of the cluster, $E_{\text{th}}/L_X$. Compare this to the age of the universe of roughly $13 \, \text{Gyrs} = 1.3 \times 10^{10} \, \text{yrs}$.

Note: The assumption of pure bremsstrahlung emission for a cluster is too simple. Thermal gas in collisional ionization equilibrium emits a lot of energy in emission lines, which complicates matters significantly (we will discuss this later in the semester). For proper research, you will need to consult a thermal emission model to get accurate formulae for the emissivity.

2 The Pressure of a Radio Galaxy

In this problem, you will investigate synchrotron emission from a radio galaxy. A radio galaxy at a distance of $16 \, \text{Mpc}$ emits diffuse synchrotron emission from a roughly circular region of about 10 arcmin in diameter. The total flux from the source at 300 MHz is measured to be $10^5 \, \text{Jy}$, where $1 \, \text{Jy} = 10^{-23} \, \text{ergs cm}^{-2} \, \text{s}^{-1} \, \text{Hz}^{-1}$ with a powerlaw index of roughly $\alpha = 0.5$ such that

$$F_{\nu} \propto \nu^{-\alpha}$$

(2.4)

From class, recall that an isotropic powerlaw distribution of electrons of the form

$$\frac{dN}{d\gamma} = N_0 \gamma^{-s} \quad \text{for} \quad \gamma_{\text{min}} \leq \gamma \leq \gamma_{\text{max}}$$

(2.5)

produces powerlaw emission with index $\alpha = (s - 1)/2$. Assume that the electrons span a wide range in $\gamma$, from some $\gamma_{\text{min}} \gg 1$ to $\gamma_{\text{max}} \gg \gamma_{\text{min}}$. Outside of this range, $dN/d\gamma = 0$. Take $dN/d\gamma$ to be the volume number density of electrons per unit interval in $\gamma$.

A) Assume the emission is coming from a uniformly emitting sphere. Determine the volume emission coefficient $\epsilon_{\nu} = dE/d\nu \, dV \, dt$ in units of $\text{ergs cm}^{-3} \, \text{s}^{-1} \, \text{Hz}^{-1}$ from the observed flux.

B) Knowing that this is synchrotron emission, determine the particle spectral index $s$ from the observed value of $\alpha$ and use it throughout the rest of this problem.

C) Given equation 2.5 above for the powerlaw electron distribution, derive an expression for the electron energy density $u_e$ in terms of $N_0$, $\gamma_{\text{min}}$, and $\gamma_{\text{max}}$. Recall that the electron energy is $\gamma m_e c^2$. Express $N_0$ in terms of $u_e$, $\gamma_{\text{min}}$ and $\gamma_{\text{max}}$. You will notice that this expression is very insensitive to the actual values of $\gamma_{\text{min}}$ and $\gamma_{\text{max}}$. 

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D) Recall that the magnetic energy density is \( u_B = B^2 / 8\pi \). Using your expression for \( N_0 \) from part C, rewrite the synchrotron emission coefficient \( dP/d\nu \) from equation 4.53 in the notes in terms of \( u_e \) and \( u_B \) instead of \( N_0 \) and \( B \).

E) Express the total energy density of the plasma \( u_{\text{tot}} = u_e + u_B \) in terms of the emission coefficient \( \epsilon_\nu \) and the ratio of magnetic to particle energy density \( R \equiv u_B / u_e \). Show that there is a value \( R_{\text{min}} \) for which \( u_{\text{tot}} \) is minimized. This is called the minimum-energy condition and is often used in analyzing synchrotron emission. Note: instead of minimum energy, one also often uses the so-called equipartition assumption, which is simply \( R = 1 \). The two assumptions are not very different.

F) Using the value for \( R_{\text{min}} \) you derived in part E, derive the minimum value for \( u_{\text{tot}} \) for the observed radio nebula. For the numerical evaluation, you may use \( \gamma_{\text{max}} = 10^3 \gamma_{\text{min}} \) and take \( \Gamma(s/4 + 19/12)\Gamma(s/4 - 1/12) \approx 2 \). Also, you may assume that the magnetic field is perpendicular to the line of sight, such that \( \sin \alpha = 1 \) in equation 4.53 in the notes.

G) Again using \( R_{\text{min}} \) from above, what is the total internal particle energy of the radio emitting gas? What is its magnetic pressure? Using the relation that \( u_e = 3p_e \) for the relativistic electrons, what is the total pressure of the radio emitting gas?