The ionization state of a gas in equilibrium at temperature $T$ can be found using Saha’s equation,

\[
\frac{N_{j+1}}{N_j} = \frac{2Z_{j+1}}{n_e Z_j} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_j/kT}.
\]  

(1)

There is a separate Saha equation for each pair $(j,j+1)$ of ionization states associated with each element present in the gas. These equations have to be solved simultaneously, together with an equation governing charge conservation which ultimately sets the electron number density $n_e$. Generally, this is a task best left to a computer.

However, the simplest case of a pure hydrogen gas is amenable to analytic solution, since there is only one Saha equation to solve. Assuming most of the neutral hydrogen is in the ground state, we can make the approximation $Z_1 \approx g_1 = 2$ (here, $g_1$ is the statistical weight of the ground state). For the ionized hydrogen, $Z_{ii} = 1$ because the ionized state is just a proton. Saha’s equation then becomes

\[
\frac{N_{ii}}{N_i} = \frac{1}{n_e} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-13.6 \text{ eV}/kT}.
\]  

(2)

To determine the electron number density $n_e$, we can take advantage of charge conservation. If there are $n$ hydrogen atoms/ions per unit volume, then

\[
n_e = \frac{N_{ii}}{N_i + N_{ii}} n,
\]  

(3)

since there is one electron for each H\,\text{II} ion. Then, Saha’s equation becomes

\[
\frac{N_{ii}^2}{N_i(N_i + N_{ii})} = \frac{N_{ii}^2}{(N - N_{ii})N} = \frac{1}{n} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-13.6 \text{ eV}/kT},
\]  

(4)

where $N \equiv N_i + N_{ii}$. Introducing the ionization fraction $x = N_{ii}/N$, this becomes

\[
x^2 \frac{1}{1-x} = \frac{1}{n} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-13.6 \text{ eV}/kT}.
\]  

(5)

This last equation can be recognized as a quadratic equation for $x$, which can be solved using standard methods. Based on the resulting solution, Fig. 1 plots the ionization fraction as a function of temperature for a pure hydrogen gas with a number density $n = 10^{20} \text{ m}^{-3}$ (a value typical to a stellar atmosphere). Note how a fraction $x = 1/2$ is reached at a temperature around $T \approx 10,000 \text{ K}$ — this is where the Balmer lines are strongest.

1
Figure 1: The ionization fraction $x$ of a pure hydrogen gas, plotted as a function of temperature.