Degeneracy

Breakdown of ideal gas law

- So far, assumed gas pressure follows ideal equation of state:
  \[ P_{\text{gas}} = \frac{\rho kT}{\mu m_H} \]

- However, at sufficiently high densities, and sufficiently low temperatures, ideal assumption begins to break down:
  * Number density of particles \( n \) is large
  * Typical momentum of particles is small
  * Particles concentrated in small volume of phase (momentum/position) space
  * But there’s a limit to how tightly particles can be packed in phase space: Pauli exclusion principle
  * In limit where exclusion principle is important, gas is degenerate; different equation of state

Non-relativistic degenerate gas

- Consider Maxwell-Boltzmann velocity distribution function for (non-degenerate) particles of mass \( m \) at temperature \( T \):
  \[ n_v dv = n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-mv^2/2kT} 4\pi v^2 dv \]
  (number of particles per unit volume with velocities in interval \([v, v + dv]\))

- Can also be written as momentum distribution function:
  \[ n_p dp = n \left( \frac{1}{2\pi mkT} \right)^{3/2} e^{-p^2/2mkT} 4\pi p^2 dp \]

- In limit \( T \to 0 \), particles concentrated around \( p = 0 \); all try to have lowest possible momentum & energy \( E(p) \), which is zero

- However, this can run foul of Pauli exclusion principle, since all particles in same momentum state

- In fact, lowest momentum/energy state has all particles filling up available cells of phase space, up to some maximum momentum \( p_F \) (the Fermi momentum)

\[ n_p dp = \begin{cases} 4\pi p^2 dp & p < p_F \\ 0 & p > p_F \end{cases} \]

Denominator: volume in phase space occupied by each distinct quantum state (spin gives factor of 2)

- Fermi momentum set by requirement
  \[ \int_0^{p_F} 4\pi p^2 dp = n, \]
  so that
  \[ p_F \sim n^{1/3} \]
  (henceforth, drop all unimportant factors in expressions, for simplicity)
– To obtain equation of state, recall that pressure scales as

\[ P_{\text{gas}} \sim u \]

where \( u \) is kinetic energy density

– To find \( u \), integrate over all occupied states

\[ u \sim \int_0^{p_F} E(p)n_p \, dp \]

If gas is non-relativistic, \( E(p) = p^2/2m \); hence,

\[ u \sim \int_0^{p_F} p^2 p^2 \, dp \sim p_F^5 \]

and

\[ P_{\text{gas}} \sim p_F^5 \sim n^{5/3} \]

– With \( \rho \sim n \), final result:

\[ P_{\text{gas}} \sim \rho^{5/3} \]

– Discussion:

* This is **degeneracy pressure**; comes from exclusion principle rather than thermal motions
* Independent of temperature
* Polytropic — a fully degenerate star is a poltrope
* Strictly applies at zero temperature, but becomes good approximation when average particle momenta are below \( p_F \)
* This limit equivalent to inequality

\[ kT \lesssim \frac{p_F^2}{2m} \equiv E_F \]

(\( E_F \) is **Fermi energy**). Note mass dependence; lighter particles become degenerate first
* Often see this inequality written as

\[ \frac{T}{\hbar c^2 m^{5/2}} < D \]

where \( D \) depends on \( m \)
* **Electron degeneracy** important in cores of low-mass stars, and in white dwarfs
* **Neutron degeneracy** important in neutron stars (no electrons left!)
* Degeneracy responsible for **helium flash**: when helium ignites, temperature increases (particles move faster), but pressure does not increase (because degeneracy condition above still holds); no ‘safety valve’ where star expands to cool off reactions, thus runaway burning

* **Relativistic degenerate gas**

– Similar to derivation above, but must use Einstein energy-momentum relation

\[ E^2 = p^2 c^2 + m_0^2 c^4 \]

\( (m_0 \) is rest mass)
- In relativistic limit $E \gg m_0c^2$, $E \sim p$
- Hence,
  
  $$u \sim \int_0^{p_F} p^2 \, dp \sim p_F^4$$
  
  and
  
  $$P_{\text{gas}} \sim \rho^{4/3}$$

- Electrons become relativistic in white dwarf stars as they approach the Chandrasekhar limit (limiting mass, above which there is collapse to neutron star)