21 — Equation of State [$Revision: 1.1$]

- Ideal gas
  - So far, stellar material has been treated as ideal gas.
  - Ideal **equation-of-state** (EOS) relates gas pressure P_g, temperature T & number of particles N:
    $$P_g V = NkT$$
    where V is volume (pressure is written as P_g instead of P to distinguish from radiation pressure). Slighly more useful form:
    $$P_g = nkT$$
    where $n = N/V$ is number of particles per unit volume (number density)
- Mean molecular weight
  - To relate n to mass density ρ:
    $$n = \frac{\rho}{\bar{m}}$$
    where $\bar{m}$ is average mass per particle. Introduce **mean molecular weight**:
    $$\mu = \frac{\bar{m}}{m_H}$$
    (average mass per particle, in units of hydrogen atom mass m_H). **NOTE:** terminology change here; previously, μ has been used to indicate the mean particle mass in units of grams etc.; now, it is the mean particle mass in units of m_H
  - Combine above expressions give pressure-density-temperature EOS:
    $$P_g = \frac{\rho kT}{\mu m_H}$$
    (most common form)
  - Looks simple; but μ in general depends on ionization state of gas
- Evaluating μ
  - To evaluate μ, must determine average particle mass $\bar{m}$ by adding over all particles
  - Let n_j be number density of atoms of type j, m_j be corresponding mass, and n_e be number density of electrons. Then
    $$\bar{m} = \frac{\sum_j n_j m_j + n_e m_e}{\sum_j n_j + n_e} \approx \frac{\sum_j n_j m_j}{\sum_j n_j + n_e}$$
    (electron masses are negligible)
  - But m_j determined from **mass number** A_j (number of protons + neutrons)
    $$m_j \approx A_j m_H$$
    (binding mass is negligible)
  - So:
    $$\mu = \frac{\bar{m}}{m_H} = \frac{\sum_j n_j A_j}{\sum_j n_j + n_e}$$

- For neutral gas ($n_{\rm e} \to 0$):

$$\mu = \frac{\sum_j n_j A_j}{\sum_j n_j}$$

- For fully ionized gas ($n_{\rm e} \to \sum_j n_j Z_j$, where Z_j is **atomic number**):

$$\mu = \frac{\sum_j n_j A_j}{\sum_j n_j (1 + Z_j)}$$

- General cases: must determine $n_{\rm e}$ from solution of **Saha equation**!

- Mass fractions
  - To work out number densities n_j, need to know **mass fractions**
  - Mass fraction X_j is the fraction *by mass* of gas that is made up of element j. E.g., in Sun helium has a mass fraction ~ 0.28, meaning 1 g of solar material contains 0.28 g of helium
  - Always true that

$$\sum_j X_j = 1$$

  - Conventions:
    * X_j for hydrogen written as X
    * X_j for helium written as Y
    * Combined X_j for all other elements written as Z (**metallicity**); do not confuse with Z_j!
    * $X + Y + Z = 1$
  - Number densities from mass fractions:

$$n_j = \frac{\rho}{m_{\rm H}} \frac{X_j}{A_j}$$

  - So, for neutral gas:

$$\mu = \frac{\sum_j X_j}{\sum_j \frac{X_j}{A_j}} = \left[\sum_j \frac{X_j}{A_j} \right]^{-1} = \left[X + \frac{Y}{4} + \left\langle \frac{1}{A_j} \right\rangle Z \right]^{-1}$$

    where $\langle 1/A_j \rangle$ is an average over metals ($\sim 1/15.5$ for solar composition)
  - For fully ionized gas:

$$\mu = \left[\sum_j \frac{X_j}{A_j} (1 + Z_j) \right]^{-1} \approx \left[2X + \frac{3Y}{4} + \frac{Z}{2} \right]^{-1}$$

    (used approx: for each metal, assume $(1 + Z_j)/A_j \approx 1/2$)

- Solar abundances
  - Sun has $X = 0.7$, $Y = 0.28$, $Z = 0.02$
  - For neutral gas, $\mu = 1.30$
  - For ionized gas, $\mu = 0.62$