Focus now changes from the surface, atmosphere layers of a star, over to the interior layers: stellar structure.

Hydrostatic equilibrium is one of the fundamental principles of stellar structure.

Along the same lines as derivation in Notes 13:

- Star supports itself against downward pull of gravity by pressure gradients
- Consider slab inside star, extending from radius \( r \) to radius \( r + dr \) (i.e., thickness is \( dr \)), and with horizontal area \( dA \)
  - Upward pressure force on inner side is \( P(r) \, dA \)
  - Downward pressure force on upper side is \( P(r + dr) \, dA \)
  - Net pressure force is \( F_P = P(r) \, dA - P(r + dr) \, dA \)
  - Net gravitational force is likewise \( F_g = -g \rho \, dr \, dA \)

For static balance:

\[
F_P + F_g = P(r) \, dA - P(r + dr) \, dA - g \rho \, dr \, dA = 0
\]

Divide through by \( dr \), \( dA \), take limit \( dr \to 0 \):

\[
\frac{dP}{dr} = -\rho g
\]

This is equation of hydrostatic equilibrium — relates pressure gradient to local density and gravitational acceleration.

Gravitational field

- Obtain \( g \) from gravitational potential \( \Phi \):
  \[
g = \frac{d\Phi}{dr}
\]

Gravitational potential comes from solution of Poisson’s equation:

\[
\nabla^2 \phi = 4\pi G \rho
\]

Special-case solution: in spherical symmetry,

\[
g = \frac{GM_r}{r^2}
\]

where \( M_r \) is total mass contained by the sphere with radius \( r \)

Important: note that expression for \( g \) does not imply that \( \Phi = -GM_r/r \) (applies only when there is no matter outside \( r \))

Mass distribution

- To find \( M_r \), just add up mass contained within sphere:
  \[
  M_r = \int_0^r 4\pi r'^2 \rho(r') \, dr'
  \]
- $M_r$ varies between 0 (core) and $M$ (surface)
- Often, useful to use $M_r$ instead of $r$ as coordinate for describing position within star

**Timescales**

- Suppose pressure could be turned off. Apply Newton’s second law to find approximate timescale for stellar collapse:

$$M \frac{R}{t^2} \sim \frac{GM^2}{R^2}$$

Solving, **freefall timescale** is

$$t_{ff} = \sqrt{\frac{R^3}{GM}}$$

- Likewise, suppose gravity could be turned off. Apply Newton’s second law to find approximate timescale for stellar explosion:

$$M \frac{R}{t^2} \sim PR^2$$

Solving, **explosion timescale** is

$$t_{ex} = \sqrt{\frac{M}{PR}}$$

- For star in hydrostatic equilibrium,

$$\frac{P}{R} \sim \frac{GM}{R^2} \frac{M}{R^3}$$

(first term on rhs is gravity, second term is density), and so

$$t_{ex} = \sqrt{\frac{M R}{R^2 P}} = \sqrt{\frac{R^3}{GM}} = t_{ff}$$

(i.e., freefall and explosion timescales are equal)

- **Dynamical timescale**: single timescale for star in hydrostatic equilibrium:

$$t_{dyn} = t_{ex} = t_{ff} = \sqrt{\frac{R^3}{GM}}$$

Dynamical timescale is typically $\sim 1$ hour for main-sequence star; indicates how long it will take for star to correct any departures from hydrostatic equilibrium.