10 — Radiative Transfer [$Revision : 1.2$]

- Transfer through absorbing medium
  - From previous notes (10): number of photons in beam N_p obeys differential equation

    $$\frac{\mathrm{d}N_\mathrm{p}}{\mathrm{d}s} = -N_\mathrm{p} n\sigma = -N_\mathrm{p}\kappa\rho$$

    where s is distance traveled in direction of propagation
  - Same equation applies to specific intensity:

    $$\frac{\mathrm{d}I_\lambda}{\mathrm{d}s} = -I_\lambda \kappa_\lambda \rho$$

    (note: now wavelength/frequency dependent; κ_λ is **monochromatic opacity**)
  - This is equation of radiative transfer for a purely absorbing medium
- Transfer through emitting medium
  - Consider radiation traveling through same slab with (infinitesimal) thickness ds, in which only emission processes take place
  - Change in specific intensity traveling through slab is

    $$\mathrm{d}I_\lambda = j_\lambda \rho \mathrm{d}s$$

    where j_λ is **emissivity** (or **emission coefficient**): amount of radiation emitted per second, per unit wavelength interval, per unit mass, per unit solid angle, in certain direction.
  - Rearranging, in limit d$s \to 0$:

    $$\frac{\mathrm{d}I_\lambda}{\mathrm{d}s} = j_\lambda \rho$$
  - This is equation of radiative transfer for a purely emitting medium
- Transfer through general medium
  - Combining above equations for medium with absorption & emission:

    $$\frac{\mathrm{d}I_\lambda}{\mathrm{d}s} = -\kappa\rho I_\lambda + j_\lambda \rho$$

    This is full **radiative transfer equation** (RTE)
  - Often written in the form

    $$-\frac{1}{\kappa_\lambda \rho}\frac{\mathrm{d}I_\lambda}{\mathrm{d}s} = I_\lambda - S_\lambda,$$

    where $S_\lambda \equiv j_\lambda/\kappa_\lambda$ is the **source function**
  - Introduce wavelength-dependent optical depth by

    $$\mathrm{d}\tau_\lambda = -\kappa_\lambda \rho \mathrm{d}s$$

    NOTE: sign difference, compared to notes 10 — convention is that optical depth increases back along ray, because we look into stars from outside (outer layers of star have $\tau_\lambda = 0$)
  - Then, RTE is

    $$\frac{\mathrm{d}I_\lambda}{\mathrm{d}\tau_\lambda} = I_\lambda - S_\lambda$$

- Formal solution of the RTE
  - Consider radiation passing through slab of material in which opacity and emissivity are known functions of s
  - Solve RTE through use of integrating factor:

$$\frac{\mathrm{d}I_\lambda}{\mathrm{d}\tau} - I_\lambda = -S_\lambda$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}\left(I_\lambda \mathrm{e}^{-\tau_\lambda}\right) = -S_\lambda \mathrm{e}^{-\tau_\lambda}$$

    Integrate from front of slab ($s = 0$) to s:

$$I_\lambda \mathrm{e}^{-\tau_\lambda} = I_{\lambda,0}\mathrm{e}^{-\tau_{\lambda,0}} + \int_{\tau_\lambda}^{\tau_{\lambda,0}} S_\lambda \mathrm{e}^{-t}\mathrm{d}t$$

    where $I_{\lambda,0}$ and $\tau_{\lambda,0}$ are intensity and optical depth at front of slab (recall: optical depth decreases as we travel through slab), and t is dummy integration variable (not time!). Rearrange:

$$I_\lambda = I_{\lambda,0}\mathrm{e}^{\tau_\lambda - \tau_{\lambda,0}} + \int_{\tau_\lambda}^{\tau_{\lambda,0}} S_\lambda \mathrm{e}^{\tau_\lambda - t}\mathrm{d}t$$

    Allows us to calculate radiation field once we know source function S_λ as a function of optical depth
  - Looks simple; but in many situations S_λ (and κ_λ) depend on I_λ!
- Simple RTE cases
  - Absorption, no emission: $S_\lambda \to 0$:

$$I_\lambda = I_{\lambda,0}\mathrm{e}^{-\Delta\tau}$$

    where

$$\Delta\tau = \tau_{\lambda,0} - \tau_\lambda$$

    Interpretation: intensity at s is incoming intensity attenuated by factor $\mathrm{e}^{-\Delta\tau}$
  - Emission, no absorption: $\kappa_\lambda \to 0$:

$$I_\lambda = I_{\lambda,0} + \int_0^s j_\lambda \rho \mathrm{d}s$$

    Interpretation: intensity at s is incoming intensity plus sum of contributions from emitting material in interval $(0, s)$
  - Constant source function:

$$I_\lambda = I_{\lambda,0}\mathrm{e}^{-\Delta\tau} + S_\lambda\left(1 - \mathrm{e}^{-\Delta\tau}\right)$$

    Interpretation: intensity at s is incoming intensity attenuated by factor $\mathrm{e}^{-\Delta\tau}$, plus contributions from emitting material (also attenuated).
  - Note: In limit $\Delta\tau \gg 1$, $I_\lambda \approx S_\lambda$
  - Homogeneous radiation field:

$$I_\lambda = I_{\lambda,0} = S_\lambda$$

    Blackbody is special case: since $I_\lambda = B_\lambda$ (Planck function), therefore

$$S_\lambda = B_\lambda$$

    and

$$j_\lambda = \kappa B_\lambda$$

    (most emission at wavelengths where opacity is high; good absorber is also good emitter)