9 — Describing a Radiation Field [$Revision: 1.3$]

- Specific Intensity
  - To fully describe a **radiation field**, we need to specify how much energy
    * At each point in space
    * At each point in time
    * In each direction
    * At each wavelength
  - All this information encapsulated in **specific intensity** I_λ
  - For an infinitessimal area element $\mathrm{d}A$, the amount of energy passing through $\mathrm{d}A$ at an angle θ to the normal, within the wavelength interval $(\lambda, \lambda + \mathrm{d}\lambda)$, within the truncated cone with solid angle $\mathrm{d}\Omega$, and within the time interval $\mathrm{d}t$ is:

$$E_\lambda \, \mathrm{d}\lambda = I_\lambda \, \mathrm{d}A \cos\theta \, \mathrm{d}\lambda \, \mathrm{d}\Omega \, \mathrm{d}t$$

  - Aside: **solid angle** is 3-dimensional analog to planar angle.
    * For circle of radius r, segment with angle ϕ (in radians) has arc length

$$\mathrm{d}s = r\,\phi$$

    * For sphere of radius r, cone with solid angle $\mathrm{d}\Omega$ (in **steradians**) has base area

$$\mathrm{d}A = r^2 \mathrm{d}\Omega$$

    * Full circle has total angle 2π rad; full sphere has 4π sterad
    * In spherical-polar coordinates, solid angle differential $\mathrm{d}\Omega$ can be written in terms of θ, ϕ differentials

$$\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

- Mean intensity & energy density
  - At each point in space, define **mean intensity** by averaging over all solid angles

$$\langle I_\lambda \rangle = \frac{1}{4\pi} \int I_\lambda \mathrm{d}\Omega = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} I_\lambda \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

  - For axisymmetric radiation field (I_λ not depending on ϕ),

$$\langle I_\lambda \rangle = \frac{1}{2} \int_0^{\pi} I_\lambda \sin\theta \mathrm{d}\theta.$$

    This often written

$$\langle I_\lambda \rangle = \frac{1}{2} \int_{-1}^{1} I_\lambda \mathrm{d}\mu$$

    where $\mu \equiv \cos\theta$ (and $\mathrm{d}\mu = \sin\theta \mathrm{d}\theta$)
  - For isotropic radiation field, $\langle I_\lambda \rangle = I_\lambda$
  - Mean intensity is related to the **specific energy density** u_λ via

$$u_\lambda \mathrm{d}\lambda = \frac{4\pi}{c} \langle I_\lambda \rangle \mathrm{d}\lambda$$

- For blackbody (recall from notes 3):

$$u_\lambda \mathrm{d}\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \mathrm{d}\lambda$$

  Since BB radiation field is isotropic,

$$I_\lambda \mathrm{d}\lambda = \langle I_\lambda \rangle \mathrm{d}\lambda = \frac{c}{4\pi} u_\lambda \mathrm{d}\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \mathrm{d}\lambda$$

  This last equation defines **Planck function** — special name for specific intensity of BB radiation field:

$$B_\lambda(T) \equiv \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

  (B for blackbody!)

- Total (bolometric) energy density for BB found by integrating over all wavelengths:

$$u = \int_0^\infty u_\lambda \mathrm{d}\lambda = \int_0^\infty B_\lambda \mathrm{d}\lambda = aT^4$$

  where $a = 4\sigma/c$ is **radiation constant**

• Flux

  - Total energy passing through surface, per second, per unit area, per unit wavelength, in all directions, defines **specific flux** F_λ (aka the monochromatic flux)
  - To find F_λ, integrate equation for energy passing through dA over all solid angles:

$$F_\lambda = \frac{\int E_\lambda}{\mathrm{d}A \mathrm{d}t \mathrm{d}\lambda} = \int I_\lambda \cos\theta \mathrm{d}\Omega$$

    Using definition of differential solid angle:

$$F_\lambda = \int_0^{2\pi} \int_0 I_\lambda \cos\theta \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

  - For axisymmetric radiation field,

$$F_\lambda = 2\pi \int_0^\pi I_\lambda \cos\theta \sin\theta \mathrm{d}\theta = 2\pi \int_{-1}^1 I_\lambda \mu \mathrm{d}\mu$$

  - Often, split flux for axisymmetric field into up ($\mu > 0$) and down ($\mu < 0$) components:

$$F_\lambda = F_{\lambda,+} - F_{\lambda,-}$$

    where

$$F_{\lambda,+} = 2\pi \int_0^1 I_\lambda \mu \mathrm{d}\mu,$$

$$F_{\lambda,-} = -2\pi \int_{-1}^0 I_\lambda \mu \mathrm{d}\mu$$

  - For isotropic radiation field,

$$F_\lambda = 2\pi I_\lambda \int_{-1}^1 \mu \mathrm{d}\mu = 0$$

    (and $F_{\lambda,+} = -F_{\lambda,-}$)

- Radiation pressure
  - Pressure can be thought of as amount of **momentum** in direction normal to surface, passing through unit area of surface each second
  - Photons carry momentum: $p = E/c$
  - Apply to radiation field: for energy E_λ passing through surface per second per unit area, momentum in normal direction is

$$p_{\lambda,\perp} = \frac{E_\lambda}{c}\cos\theta$$

  - Integrate over all solid angles to get **radiation pressure**

$$P_{\mathrm{rad},\lambda} = \frac{\int E_\lambda \cos\theta}{c\mathrm{d}A\mathrm{d}t\mathrm{d}\lambda} = \frac{1}{c}\int I_\lambda \cos^2\theta \mathrm{d}\Omega$$

  - **Note:** the radiation pressure exists irrespective of whether the photons interact with matter. There is net force due to radiation when there is a **gradient** of radiation pressure (i.e., imbalance in radiation pressure on opposite sides of region).
  - For axisymmetric radiation field,

$$P_{\mathrm{rad},\lambda} = \frac{2\pi}{c}\int_0^\pi I_\lambda \cos^2\theta \sin\theta \mathrm{d}\theta = \frac{2\pi}{c}\int_{-1}^{1} I_\lambda \mu^2 \mathrm{d}\mu$$

  - For isotropic radiation field,

$$P_{\mathrm{rad},\lambda} = \frac{2\pi}{c} I_\lambda \int_{-1}^{1} \mu^2 \mathrm{d}\mu = \frac{4\pi}{3c} I_\lambda$$

  - Total radiation pressure (integrated over all wavelengths):

$$P_{\mathrm{rad}} = \int_0^\infty P_{\mathrm{rad},\lambda} \mathrm{d}\lambda$$

    For isotropic,

$$P_{\mathrm{rad}} = \frac{1}{3}u$$

    (compare with monatomic gas: $P = 2/3u$)

‘