Assignment 4 — Solutions [Revision : 1.2]

1. (a) Rearranging the first-moment equation,

\[ F_{\lambda} = -\frac{c}{\kappa \rho} \frac{dP_{\text{rad,}\lambda}}{dr} \]

Substituting in the expression for \( P_{\text{rad,}\lambda} \), this becomes

\[ F_{\lambda} = -\frac{4\pi}{3\kappa \rho} \frac{dB_{\lambda}}{dr}. \]

The chain rule is used to write

\[ \frac{dB_{\lambda}}{dr} = \frac{dB_{\lambda}}{dT} \frac{dT}{dr}, \]

from which we obtain the result

\[ F_{\lambda} = -\frac{4\pi}{3\kappa \rho} \frac{dB_{\lambda}}{dT}. \]

(b) Integrating over all wavelengths,

\[ F = \int_{0}^{\infty} F_{\lambda} d\lambda = -\int_{0}^{\infty} \frac{4\pi}{3\kappa \rho} \frac{dB_{\lambda}}{dT} \frac{dT}{dr} d\lambda. \]

Bringing the wavelength-independent terms out from under the integral sign,

\[ F = -\frac{4\pi}{3\rho} \frac{dT}{dr} \int_{0}^{\infty} \frac{1}{\kappa} \frac{dB_{\lambda}}{dT} d\lambda. \]

(c) Equating the two expressions for \( F \),

\[ \frac{4\pi}{3\rho} \frac{dT}{dr} \int_{0}^{\infty} \frac{1}{\kappa} \frac{dB_{\lambda}}{dT} d\lambda = \frac{4acT^3}{3\kappa \rho} \frac{dT}{dr}. \]

Rearranging,

\[ \frac{1}{\bar{\kappa}} = \frac{\int_{0}^{\infty} \frac{1}{\kappa} \frac{dB_{\lambda}}{dT} d\lambda}{acT^3 / \pi}. \]

Recognizing that

\[ \int_{0}^{\infty} B_{\lambda} d\lambda = \frac{\sigma T^4}{\pi} = \frac{acT^4}{4\pi}, \]

the expression for \( \bar{\kappa} \) may also be written

\[ \frac{1}{\bar{\kappa}} = \frac{\int_{0}^{\infty} \frac{1}{\kappa} \frac{dB_{\lambda}}{dT} d\lambda}{\int_{0}^{\infty} \frac{dB_{\lambda}}{dT} d\lambda}. \]

highlighting the fact that \( 1/\bar{\kappa} \) is a weighted average of \( 1/\kappa_{\lambda} \).

2. (a) Because

\[ R = \lambda_{n} \xi_{1}, \]

it follows that

\[ \lambda_{n} = \frac{1.01 \times 10^{10} \text{ cm}}{6.897} = 1.01 \times 10^{10} \text{ cm} \]
(b) Using the equation

\[ M = -4\pi \lambda^2_n \rho c \xi^2_1 \left. \frac{dD_n}{d\xi} \right|_{\xi_1}, \]

with \( M = 1 \, M_\odot \), it follows that

\[ \rho_c = 76.3 \, \text{g cm}^{-3}. \]

(c) To find the central pressure, we use the polytropic relation:

\[ P = K \rho^n = K \rho^{1/n+1}. \]

The constant \( K \) follows from the relation

\[ \lambda_n = \left( n + 1 \right) \left( \frac{K \rho_c^{1/n-1}}{4\pi G} \right) \gamma^{1/2}; \]

rearranging,

\[ K = \frac{4\pi G \lambda^2_n}{\rho_c^{1/n-1} (n + 1)} \]

and so

\[ K = 3.84 \times 10^{14} \]

(with funny units). Now using the polytropic relation,

\[ P_c = 1.24 \times 10^{17} \, \text{dyne cm}^{-2}. \]

(d) If the radiation pressure is negligible, the gas obeys the ideal equation of state,

\[ P = \frac{\rho kT}{\mu m_H} \]

Solving for the central temperature,

\[ T_c = 1.21 \times 10^7 \, \text{K} \]

3. From the various equations relating the mass and radius of polytropes, we find

\[ R \propto \lambda_n \]

but

\[ \lambda_n \propto K^{1/2} \rho_c^{1/2n-1/2}. \]

Due to the degenerate equation of state, \( K = A \) is a constant — we don’t have the liberty to choose it. Also, since \( \gamma = 5/3 \), it follows that \( n = 1/(\gamma - 1) = 3/2 \). Hence,

\[ R \propto \rho_c^{-1/6} \]

But

\[ M \propto \lambda^3_n \rho_c \propto R^3 R^{-6} \]

and so

\[ R \propto M^{-1/3}. \]

4. See Fig. 1 for the plot. The gradient is estimated by eye as \( \frac{d\log P}{d\log \rho} \equiv \gamma \approx 10/8 \), from which we obtain \( n = 1/(\gamma - 1) \approx 4 \).

5. See Fig. 3 for the plot.

6. See Fig. 3 for the plot. The two luminosities differ in the core of the star \((r/R \lesssim 0.25)\), with \( L_{\text{rad}} < L_r \), because there a fraction of the total luminosity \( L_r \) is transported by convection rather than radiation.
Figure 1: Pressure-density plot for the 10 $M_\odot$ model (Q4).
Figure 2: Temperature gradient plot for the $10\,M_\odot$ model (Q5).
Figure 3: Luminosity plot for the 10 $M_\odot$ model (Q6). The solid line shows the radiative luminosity $L_{\text{rad}}$, and the dotted line the total interior luminosity $L_r$. 