A Novel Approach to Solving the Linearized Stellar Pulsation Equations

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We outline a novel approach to solving the linearized equations of stellar pulsation, which avoids pathologies encountered in previous formulations. Our approach is implemented in GYRE, a new pulsation code applicable to differentially rotating stars. To demonstrate the strengths of the code, we apply it to calculate the frequency spectrum and eigenfunctions of g modes in a white dwarf model; and to explore the non-uniform splitting of a mode multiplet in a rotating B-type main sequence model.

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Introduction

The process of solving the linearized stellar pulsation equations lies at the heart of a number of areas of current astrophysical interest – e.g., stellar stability, angular momentum transport, tidal interactions, and most prominently, asteroseismology. The pulsation equations pose a boundary-value problem (BVP), in which boundary conditions at the stellar core and surface can be satisfied only for certain choices of the pulsation frequency ω; that is, ω is an eigenvalue of the system.

Previous techniques for solving the pulsation BVP have almost universally suffered from what we call *premature normalization* – the adoption of a specific normalization choice (i.e., imposing a certain solution value at a certain location) during the search for eigenfrequencies. An example is the method popularized by Castor [1]: one of the boundary conditions is set aside, allowing solutions to be obtained that satisfy the remaining boundary conditions and normalization condition for arbitrary ω. The eigenfrequencies are then found by varying ω until the discarded boundary condition is also satisfied; if this boundary condition is represented by the equation $D(\omega)=0$, then the eigenfrequencies are the roots of this equation.

Castor's method has been used by many authors, within the computational frameworks of both relaxation (finite differencing) (e.g., [2]) and shooting (e.g., [3]). However, it has the drawback that the discriminant function $D(\omega)$ can be pathological, often exhibiting singularities. The latter arise from the imposition of the normalization condition at locations where the solution exhibits a zero. Because these locations cannot be predicted a priori, it is very difficult to find a normalization that does not cause an ill-behaved $D(\omega)$.

Our approach to solving the pulsation equations does not require any choice of normalization during the search for eigenfrequencies. Instead, we cast the equations and all accompanying boundary conditions (but *not* a normalization condition) as a linear, homogeneous matrix equation $\mathbf{S}(\omega)\,\boldsymbol{y} = 0$. Non-trivial solutions $\boldsymbol{y} \neq \mathbf{0}$ are found when the determinant of $\mathbf{S}$ vanishes. Thus, the characteristic equation, whose roots define the pulsation eigenfrequencies, becomes $D(\omega) = \det[\mathbf{S}(\omega)] = 0$.

The GYRE Code

We have implemented this approach in a new code, GYRE, which solves the equations governing adiabatic pulsation in a differentially rotating stellar model. The system matrix $\mathbf{S}(\omega)$ is set up using a multiple shooting technique [4]. To shoot across each shell of the model, we (reasonably) assume that the equation coefficients are constant within the shell. The integration can then be undertaken semi-analytically with the aid of matrix exponentials. The advantage of this approach, which is similar to that in [5], is that it permits arbitrary spatial resolution. Therefore, regions where the eigenfunctions have very short wavelengths pose no especial difficulty to GYRE.

Although GYRE remains under active development, it already shows great promise in terms of its flexibility and robustness. The following sections present example applications demonstrating different strengths of the code.

Mode Spectrum of a White Dwarf

Seismic analyses of white dwarfs (WDs) can be challenging, because the Brunt-Väisälä frequency is real throughout the degenerate core, leading to a large trapping cavity and a dense spectrum of gravity (g) modes. Fig. 1 shows the frequency spectrum for a WD model ($T_{\rm eff}$ = 11,500 K, M = 0.6 $M_\odot$, log g = 8.0; kindly provided by Mike Montgomery), for g modes having harmonic degrees in the range $1 \leq \ell \leq 200$. This spectrum comprises almost 4,000 eigenfrequencies, yet took minimal effort to compute (~4 minutes on a multicore workstation; we note here that GYRE is parallelized using MPI).

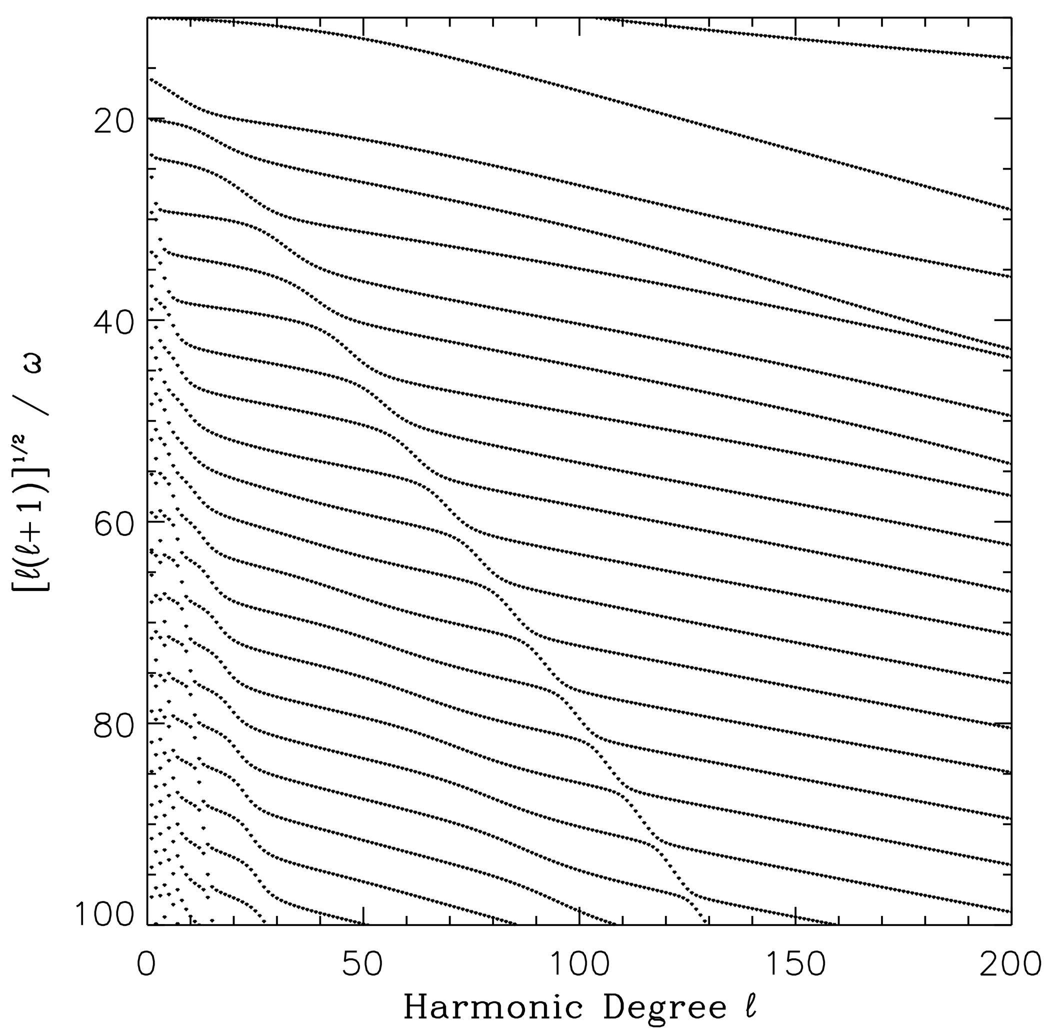

Fig. 1: Frequency spectrum of the WD model, showing the quantity $[\ell(\ell+1)]^{1/2}$ / ω (which scales approximately proportionally to the mode radial order n) as a function of harmonic degree ℓ for each mode. Note the avoided crossings in the lower center of the figure. Here and throughout, frequencies are normalized by $(GM/R^3)^{1/2}$.

A High-Resolution Eigenfunction

To demonstrate the arbitrary spatial resolution that GYRE can achieve, Fig. 2 shows the relative radial displacement eigenfunction $\delta r/R$ for a high-order g mode of the same WD model considered above. A key point to note here is that although hundreds of thousands of points are required to resolve the eigenfunction fully, accurate calculation of the corresponding *eigenfrequency* ω requires only the ~ 430 original shells of the model! This is thanks to the use of matrix exponentials.

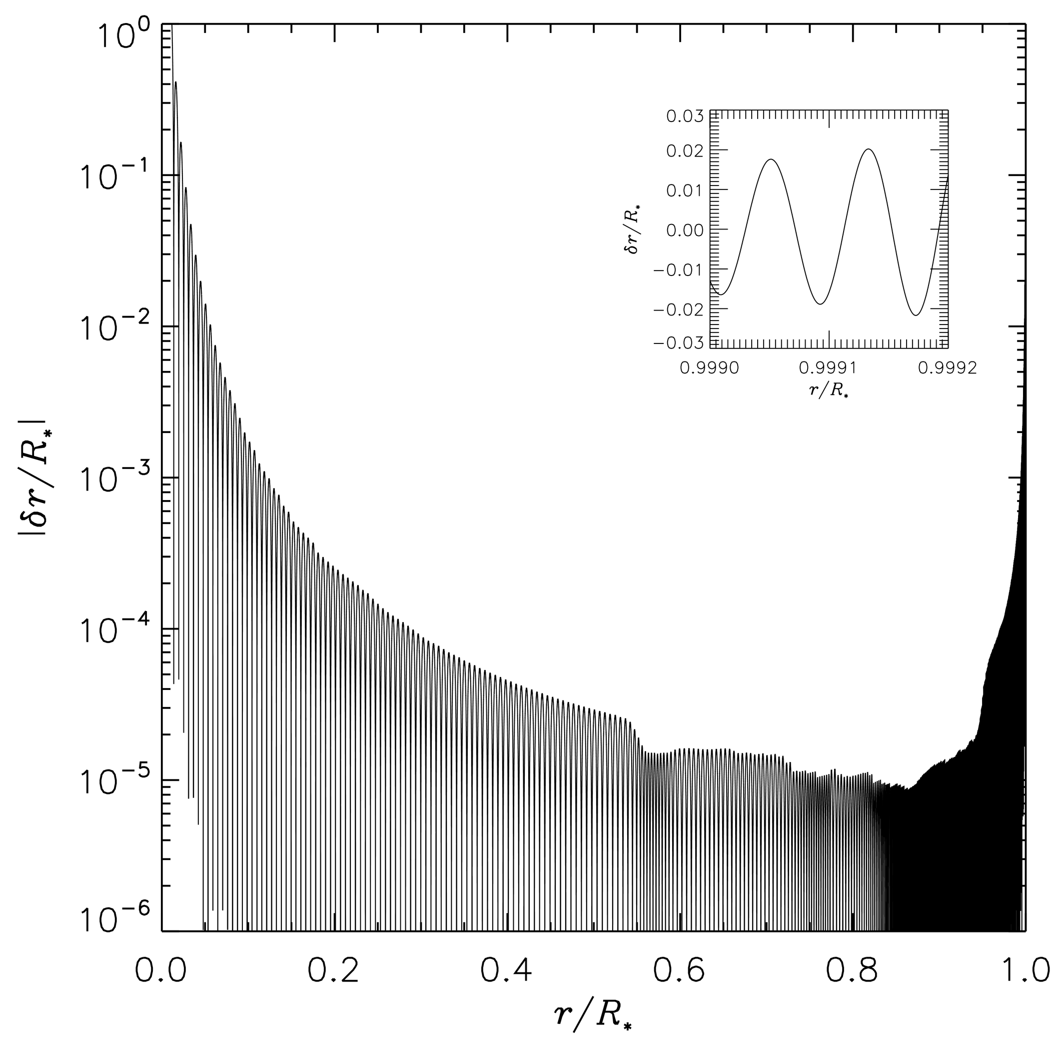

Fig. 2: Modulus of the displacement eigenfunction for the $\ell = 2$, n = 700 g mode of the WD model, plotted as a function of fractional radius on a logarithmic scale. The inset illustrates the very short radial wavelength of the eigenfunction just below the stellar surface.

Rotational Splitting in a B star

GYRE treats the Coriolis and (non-deformational) centrifugal effects arising from arbitrary differential rotation using the spherical harmonic expansion described by [6]. As a simple initial demonstration, Fig. 3 shows the effects of varying *uniform* rotation angular frequency Ω on the frequencies of an $\ell = 2$, $-2 \leq m \leq 2$ mode quintuplet of a mid-B type main sequence stellar model ($T_{\rm eff}$ = 15,200 K, M = 5.0 $M_\odot$, log g = 3.9). Although first-order perturbation theory predicts a uniform splitting of the mode frequencies, $\Delta\omega \sim -m\Omega$, departures from this splitting can be seen in the figure. In particular, the axisymmetric (m = 0) mode shows an increase in frequency toward larger Ω; this is a direct result of second-order Coriolis effects.

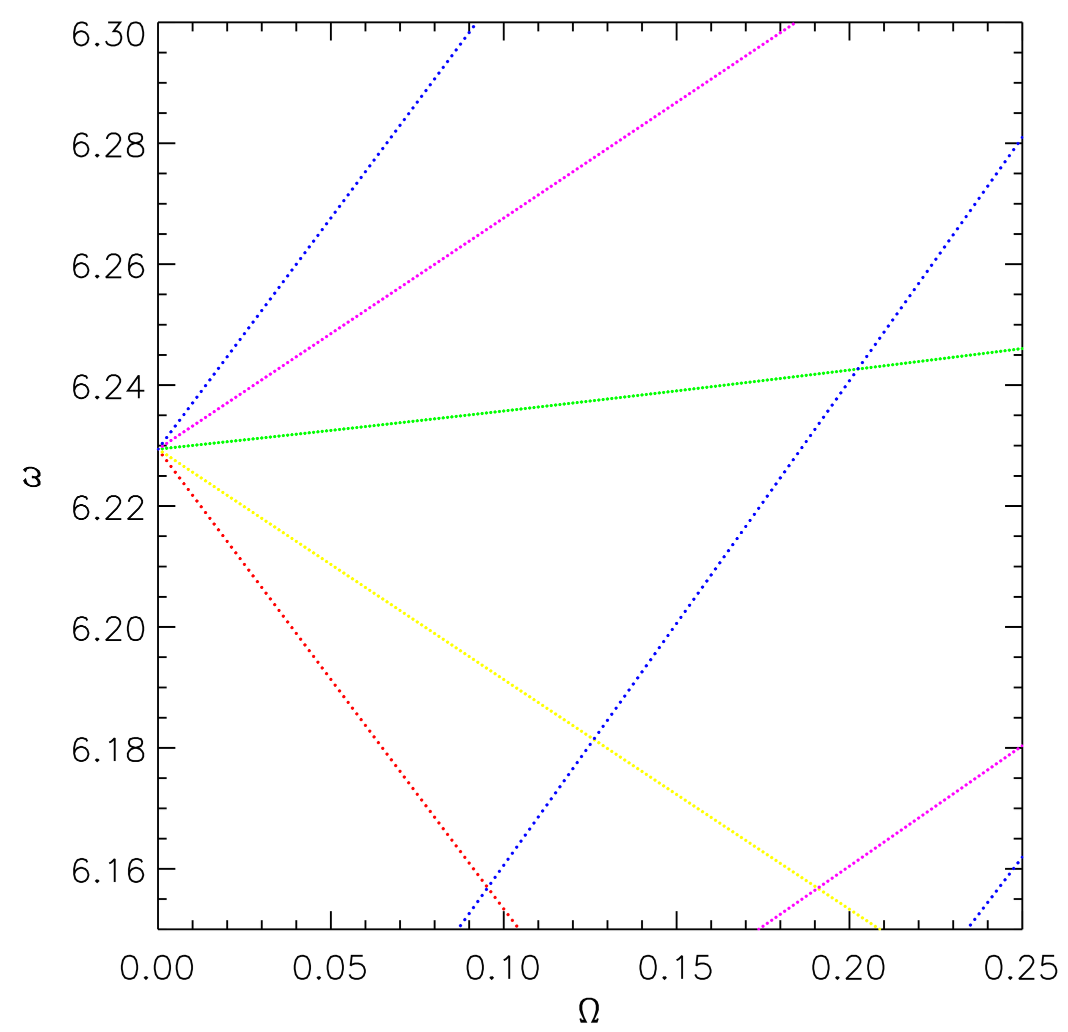

Fig. 3: Frequencies of an $\ell = 2$ mode quintuplet of the B-star model, as a function of rotation angular frequency Ω. Colors indicate the azimuthal order m: red (2), yellow (1), green (0), magenta (-1) and blue (-2). At the bottom of the figure, prograde (m < 0) modes from adjacent quintuplets can be seen.

Summary

The GYRE code uses a new approach to solving the linearized pulsation equations, which avoids the pathologies introduced by premature normalization. It enjoys arbitrarily high spatial resolution, and is applicable to differentially rotating stars. At the moment, the code is limited to adiabatic pulsation, but we are in the process of incorporating non-adiabatic processes.

Upon its full completion, GYRE will be released into the public domain under an open-source license. Until then, anyone with an interest in using the code should contact one of the authors.

References

[1] Castor, J. 1971, *ApJ*, **166**, 109; [2] Osaki, Y. & Hansen, C. 1973, *ApJ*, **185**, 277; [3] Christensen-Dalsgaard, J. 2008, *Ap&SS*, **316**, 113; [4] Ascher, U., Mattheij, R., & Russell, R. 1995, *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*; [5] Gabriel, M. & Noels, A. 1976, *A&A*, **53**, 149; [6] Unno, W., Osaki, Y., Ando, H., Saio, H., & Shibahashi, H. 1989, *Nonradial Oscillations of Stars*

We gratefully acknowledge support from NSF *Astronomy & Astrophysics Grants* #AST 0908688