The Pulsation-Rotation Interaction: Greatest Hits and the B-side

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How does rotation affect pulsation?
\[ \omega = \omega_0 + m\Omega (1 - C_{n,\ell}) \]

**Inertial Frequency**  **Unperturbed Frequency**  **Doppler Shift**  **Coriolis Perturbation**
Third-Order Perturbation Theory

P. Soufi et al.: Effects of moderate rotation on stellar pulsation. I 923

We feel perturbation theory calculations are still useful... Undoubtedly, the use of [2-D hydrocodes] will ultimately be unavoidable, but then it will be very helpful to have a code based on the perturbational approach for comparisons at moderate equatorial velocities where both are valid.”

Soufi et al. (1998)
Non-Perturbative Techniques

- Solve PDEs directly
  - Savonije et al. (1995); Clement (1998)

- Expand in $Y_{lm}$ expand, solve coupled ODEs
  - Durney & Skumanich (1968); Lee & Baraffe (1995); Reese et al. (2006); Ouazzani et al. (2012)

- Ray tracing (acoustic waves)
  - Lignières & Georgeot (2008)

- Traditional approximation (gravity waves)
  - Berthomieu et al. (1978); Bildsten et al. (1996); Lee & Saio (1997); Townsend (2003); Dziembowski et al. (2007); Mathis et al. (2008)
p-Modes in Rapidly Rotating Stars

**Polytropic**

- Island
  - M=25.0M⊙, \( \eta = 0.9 \), \( \alpha = 0.0 \)
- Chaotic
  - M=25.0M⊙, \( \eta = 0.9 \), \( \alpha = 0.0 \)
- Whispering gallery
  - M=25.0M⊙, \( \eta = 0.9 \), \( \alpha = 0.0 \)

**SCF**

- Small \( \ell - |m| \)
- Intermediate \( \ell - |m| \)
- Large \( \ell - |m| \)

Reese et al. (2009)
Ray Tracing, Poincaré Section

We found that increasing the stellar rotation leads to a soft transition from integrability to chaos analogous to the one described by the KAM theorem. As illustrated in Fig. 1 for a given rotation rate, the phase space shows the PSS and typical acoustic rays at a rotation axis [7]. The Husimi distribution is then constructed from a cut taken along the PSS: it is first scaled by the square root of the distance to the rotation axis [7]. The Husimi distribution is then compressed to compare a three-dimensional mode with the acoustic frequency. To establish a link with the asymptotic ray dynamics, we use a phase-space representation designed to assist in the observable acoustic modes.

As shown in Fig. 3, the frequency spacings of the island region form an independent subset with specific dynamical properties. The frequency spectrum thus appears as the superposition of independent frequency subsets reflecting the phase space structure. This surmise has been found to hold, and even more important to assess is supposed to hold, and even more important to assess if it is still relevant to the observable acoustic modes.

Having defined subsets of modes, we can now analyze some correlations may remain between the frequency sub-sets due to modes localized at the border between zones or due to the presence of partial barriers in phase space. This situation has been found several times in the dynamics of the stellar oscillation, and generally it was surmised that the stationary waves localized on one of these resonances is in unit of the stellar mass and the gravitational constant.

Chaotic Mode

Whispering Gallery Mode

Lignières & Georgeot 2008

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Frequency Reorganization

Fig. 5. Evolution of the frequencies of $\ell = 1, 2, 3$ modes (top to bottom). Frequencies are computed in the corotating frame. Perturbative approximations have been tested for a typical $\gamma$ Dor (left panels) and for a B star (right panels). Green/red/blue parts of curves indicate that 1st/2nd/3rd order is sufficient to reproduce complete calculations within an error $\delta \nu = 0.1 \mu Hz$. Error bars on the righthand side of each panel show $10 \times \delta \nu$.

Magenta lines indicate $\omega = 2 \Omega$. For each plot, the bottom $x$-axis and left $y$-axis show dimensionless units, whereas the top $x$-axis and right $y$-axis show physical units.

Ballot et al. (2010)
Frequency Reorganization: p-Modes

\[ \omega \approx \tilde{n}\tilde{\Delta}_n + \tilde{\ell}\tilde{\Delta}_\ell + m^2 \tilde{\Delta}_{\tilde{m}} - m\Omega_{\text{fit}} + \tilde{a} \]

Reese et al. (2009)

Even Parity

\[ \tilde{n} = 2n \]
\[ \tilde{\ell} = \frac{\ell - |m|}{2} \]

Odd Parity

\[ \tilde{n} = 2n + 1 \]
\[ \tilde{\ell} = \frac{\ell - |m| - 1}{2} \]
The Spin Parameter

\[ \nu = 2 \frac{\Omega}{\omega_c} \]

...a measure of how much the star turns during one oscillation period

**Inertial regime:** \( \nu > 1 \)
The Traditional Approximation

- Neglect the Coriolis force arising from the horizontal component of the rotation vector
- Valid when $\omega_c^2, \Omega^2 \ll N^2$ (g-modes)
- Pulsation equations identical to non-rotating case (with Cowling approx.), except:

\[
\ell(\ell + 1) \rightarrow \lambda_{\ell,m}(\nu)
\]

\[
Y_{\ell}^m(\theta, \varphi) \rightarrow \Theta_{\ell,m}(\theta; \nu) \exp(i m \varphi)
\]
V. On the Application of Harmonic Analysis to the Dynamical Theory of the Tides.—
Part II. On the General Integration of Laplace’s Dynamical Equations.

By S. S. Hough, M.A., Fellow of St. John’s College and Isaac Newton Student in
the University of Cambridge.

Communicated by Professor G. H. Darwin, F.R.S.

Received October 27,—Read December 9, 1897.
The Equatorial Waveguide

\[ \cos(\theta) = \pm \nu \]

Matsuno (1966)
Frequency Reorganization: $g$-Modes

$$\omega_c \approx \sqrt{(2\ell_\mu - 1)\Omega} \frac{W}{n}$$

Townsend (2003)

**Prograde**

$$\ell_\mu = \ell - |m|$$

**Retrograde**

$$\ell_\mu = \ell - |m| + 2$$

Note: NOT for prograde sectoral modes
Frequency Reorganization: Equatorial Kelvin Modes

$$\omega_c \approx m \frac{W_K}{n}$$

- Become prograde sectoral g-modes in non-rotating limit
- Geostrophic: $\theta$ Coriolis force balances $\theta$ pressure gradients
- Independent of rotation rate
- Azimuthally dispersion-free ($\alpha$ Oph? KIC 8054146?)
Azimuthal Dispersion Diagram

Figure 12.4: Non-dimensionalized dispersion relations for the Matsuno modes. The first two Rossby and gravity modes ($n=1$ and $n=2$) are shown. For these modes the approximate dispersion relations (12.33) and (12.34) are used. These differ only slightly from the exact formulas.

As we showed previously, shallow water modes with a characteristic speed of $c$ are isomorphic to hydrostatic gravity waves with vertical wavenumber $m$ (12.42) where $N$ is the Brunt-Väisälä frequency. For $N=1.0$ s$^{-1}$ and $c=1.6$ ms$^{-1}$, $m=6.25 \times 10^{-4}$ m. However, as a first guess, one might expect that the gravity wave would have a half-vertical wavelength equal to the depth of deep convection, which is essentially the height of the tropopause. In this way, the vertical profile of heating by the convection, which is roughly in the form of a sinusoidal function with a half-wavelength equal to the depth of the convection, would project maximally onto the vertical velocity profile of the gravity wave. In the tropics the tropopause height is about $H=16$ km, which would imply a vertical wavenumber of $m=\frac{\pi}{H}=1.96 \times 10^{-4}$ m, which is about $1/3$ of that implied by observations.

Emanuel et al. (1994) proposed a theory which addressed this problem by hypothesizing $\ell=|m|+1$ $\ell=|m|+2$ $\ell=|m|+3$ $\ell=|m|$. 

\[ \frac{\omega_r}{\sqrt{\Omega}} \alpha \frac{m}{\sqrt{\Omega}} \]
r-Modes

- Rotating versions of toroidal modes
- Propagate by conservation of vorticity
- Almost incompressible
  - No temperature perturbations
  - Difficult to excite by heat eng.
Fig. 5. Evolution of the frequencies of $\ell = 1, 2, 3$ modes (top to bottom). Frequencies are computed in the co-rotating frame. Perturbative approximations have been tested for a typical $\gamma$ Dor (left panels) and for a B star (right panels). Green/red/blue parts of curves indicate that 1st/2nd/3rd order is sufficient to reproduce complete calculations within an error $\delta \nu = 0.1 \mu \text{Hz}$. Error bars on the righthand side of each panel show $\delta \nu$ and $10 \times \delta \nu$. Magenta lines indicate $\omega = 2 \Omega$. For each plot, the bottom $x$-axis and left $y$-axis show dimensionless units, whereas the top $x$-axis and right $y$-axis show physical units.

is also extensively used to determine $g$-mode frequencies (e.g. Berthomieu et al. 1978; Lee & Saio 1997). We will also analyze how rotation affects the regularities of the spectrum – such as the period spacing – and compare it to the predictions of the perturbative and traditional methods. In the present study, we have focused on low-degree modes, but a more complete exploration clearly needs to be performed. In particular, we might look for the singular modes predicted by Dintrans & Rieutord (2000).
Rosette Modes: A Cautionary Tale

Takata & Saio (Poster #66)

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Mode Visibilities

- With rotation:
  - mode visibility generally decreases
  - often, modes are more visible from the poles
  - amplitude ratios become dependent on $m$ and $i$

- More recent studies:
  - Daszyńska-Daszkiewicz et al. (2007)
  - Reese et al. (2012)
Differential Rotation: Critical Layers

Doppler shift: \( \omega_c(r) = \omega - m\Omega(r) \)

Dispersion relation: \( k_r^2 \sim k_h^2 \frac{N^2}{\omega_c^2} \)

\[ m\Omega(r) \to \omega \]
\[ \omega_c(r) \to 0 \]
\[ k_r \to \infty \]

...the wave will be strong damped (absorbed) at the critical layer

(see poster #39)
How does pulsation affect rotation?
Angular Momentum Transport by Pulsations

“...prograde modes can carry angular momentum from wave excitation regions to wave dissipation regions, and retrograde modes can do the contrary.”

Ando (1983)
Uniform Rotation of the Sun’s Interior

...evidence for $J$ extraction by stochastic g modes?

(Talon et al. 2002)
Reynolds Decomposition of Azimuthal Momentum Equation

\[ \frac{\partial}{\partial t} \langle \bar{\omega} \bar{\rho} \bar{v}_\phi \rangle = - \frac{1}{4\pi r^2} \frac{\partial}{\partial r} L_J - \frac{\partial}{\partial t} \langle \bar{\omega} \rho' \bar{v}_\phi \rangle - \left\langle \rho' \frac{\partial \Phi'}{\partial \phi} \right\rangle \]

- Change in shell \( J \)
- Divergence of \( J \) Luminosity
- Change in wave \( J \)
- Gravitational Torque
Angular Momentum Luminosity

(angular momentum passing per unit time through spherical shell)

\[ L_J = 4\pi r^2 \langle \varpi (\bar{\rho} \bar{v}_r \bar{v}_\phi + \bar{v}_\phi \bar{\rho}' v'_r + \bar{\rho}' v'_r v'_\phi) \rangle \]

Reynolds Stress Flux

Eddy Mass Flux

Third Order Stuff
Example: $L_J$ in a Massive MS Star
Amplitude Limitation via Interaction with Rotation?

![Graph showing amplitude vs. time with unstable and stable regions marked](image-url)
Just this year…

- Rogers et al. (2013): massive stars
- Alvan & Mathis (2013): critical layers  (also, Poster #39)
- Mathis et al. (2013): interactions w/ other transport mechanisms
- Charbonnel et al. (2013): PMS stars
The pulsation-rotation interaction is tricky!
Nevertheless, much progress in past decade:
- 2-D numerical modeling becoming widespread
- Complementary tools are also emerging
- Interest in J transport is taking off
Crunch time: theory vs. observations