Again, use units in which the Plummer model parameters are \( a_P = 1 = M \), and \( G = 1 \).

1. Find the angular speed \( \Omega \) of a circular orbit in the Plummer model at radius \( r \), and the epicycle frequency \( \kappa \).

2. What is the angular momentum of a circular orbit at radius 1? Use your program to integrate an orbit with this angular momentum that starts at \( x = 1.3, y = 0 \) with \( v_x = 0 \) and \( v_y > 0 \), for about 3 cycles – it should look like Figure 3.9.

3. Now think about how to compare this orbit with what you expect from the epicycle approximation. Convert the \( x, y \) pairs describing your orbit to polar \( R, \phi \). (Why didn’t you want to do the integration in \( R, \phi \)? It’s the same reason that you can’t use WIYN to track a star near the zenith.) Now transfer to a frame rotating at the speed of the circular orbit at \( R = 1 \). This is the path of the guiding center – following a circular orbit with the same angular momentum as your particle. Plot the path of the particle in this rotating frame. It should be a near-ellipse around the guiding center. What are the minimum and maximum \( R \)-values? Which one deviates more from \( R = 1 \)? Why? (Look at Figure 3.7.)

4. Now set up several particles with guiding centers at \( R_g = 1 \) with the same epicyclic amplitude \( X \), so as to make an oval shape like Fig 5.28. At the initial time \( t = 0 \), pick e.g. 15 guiding centers evenly spaced in azimuth \( 0 \leq \phi_{gc} \leq 2\pi \). Set up the particles on their epicycles following the prescription on page 210 of the text (set \( X = 0.2 \), you get to pick \( m \)):

\[
R = R_g + X \cos[\kappa t + m\phi_{gc}] \, , \quad \phi = \phi_{gc} - \frac{2\Omega}{R_g\kappa} X \sin[\kappa t + m\phi_{gc}] \, ,
\]

where \( R = R_g + x \) as in Equation 3.68, \( \phi \) is given by Equation 3.72, and \( t = 0 \). Convert these \( R, \phi \) pairs to \( x, y \) and plot the particle positions at the start. Mark the direction of some feature (e.g. the long axis of the ellipse if you picked \( m = 2 \)).

In the Plummer potential, integrate all these particles for time \( t_{orbit} = 2\pi/\Omega(R_g) \) around the guiding center. Plot the positions of all the particles on the same plot, for \( 0, 0.1, 0.2, 0.3, 0.5 \) and 1.0) \( \times t_{orbit} \). You should see a rotating pattern. If the epicycle approximation was exact, your feature would have moved by an angle \( (\Omega - \kappa/m)t \) from its initial position. Mark this angle on each of your plots and comment on how close the epicycle approximation comes to the result of your integration.

5. Only if you are really curious about how epicycles work: set up particles on guiding centers at \( R_g = 0.8 \) and \( R_g = 1.2 \) in the same way as for part 4. (You need to change \( \Omega(R_g) \) and \( \kappa \) to be right for the new guiding center.) Integrate particle paths for all 3 rings of particles to see how the fact that the pattern speed \( (\Omega - \kappa/m) \) depends on radius causes a spiral to form.