

Astro 730 Spring 2006

Homework 3: due Friday March 31st

1) (Problem 3.19) The Sun has $v_x \approx -10 \text{ km s}^{-1}$ and $v_y \approx 5 \text{ km s}^{-1}$ relative to the Local Standard of Rest, a circular orbit at the Sun's present Galactocentric distance R_0 . Why do we know that its guiding center radius $R_g > R_0$? Assuming the Milky Way's rotation curve to be roughly flat, with $V(R) = R\Omega(R) = 200 \text{ km s}^{-1}$ and $R_0 = 8 \text{ kpc}$, find the epicyclic frequency κ . Use Equations 3.68 and 3.74 to show that $R_g \approx 8.2 \text{ kpc}$, and the extent of the Sun's radial excursions is $X = 0.35 \text{ kpc}$.

2) (Problem 5.13) Show that if the rotation curve of the Milky Way is flat near the Sun, then $\kappa = \sqrt{2}\Omega(R)$, so that locally $\kappa \approx 36 \text{ km s}^{-1} \text{ kpc}^{-1}$.

The density wave theory shows that disk stars can respond to the spiral's gravitational pull to strengthen it, only if the difference between the pattern speed Ω_p and the local circular speed $\Omega(R)$ is small enough. Specifically, the frequency $m|\Omega_p - \Omega(R)|$ with which an m -armed pattern 'bumps' the stars must be less than the epicycle frequency κ . Sketch the curves of Ω , $\Omega \pm \kappa/2$, and $\Omega \pm \kappa/4$ in a disk where $V(R)$ is constant everywhere, and show that the zone where 2-armed spiral waves can survive is much larger than that available to 4-armed spirals.

3) (Problem 3.17) The motion of a star around a non-rotating black hole of mass \mathcal{M}_{BH} is given by Equation 3.64 with an effective potential Φ_{eff} : we have

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(c^2 - \frac{2G\mathcal{M}_{\text{BH}}}{r}\right) \left(1 + \frac{L^2}{c^2 r^2}\right) \equiv E^2 - 2\Phi_{\text{eff}}(r). \quad 3.1$$

We can interpret r as distance from the center, and τ as time. (In fact r is the usual Schwarzschild radial coordinate, τ is proper time for a static observer at radius r , and E , L are the energy and angular momentum per unit mass as measured by that observer.)

To have a circular orbit at radius r , we need $dr/d\tau = d^2r/d\tau^2 = 0$. Why does this mean that Φ_{eff} has a minimum or maximum at that radius? [You can argue this one from a sketch of Φ_{eff} , or differentiate $(dr/d\tau)^2$.] If this circular orbit is stable, what does that tell us about $d^2\Phi_{\text{eff}}/dr^2$?

Differentiate Equation 3.1 to find $d\Phi_{\text{eff}}/dr$. At a circular orbit of radius r , what is L ? Show that there are no circular orbits for $r < 3G\mathcal{M}_{\text{BH}}/c^2$, since L^2 could not be positive. Sketch or plot Φ_{eff} for $L = 6G\mathcal{M}_{\text{BH}}/c$ and $L = 3G\mathcal{M}_{\text{BH}}/c$; mark the circular orbits. Which of these is stable? Explain.

Show that a circular orbit of radius r is stable only if $r > 6G\mathcal{M}_{\text{BH}}/c^2$, which requires $L > 2\sqrt{3}G\mathcal{M}_{\text{BH}}/c$.

S.L. Shapiro and S.A. Teukolsky 1983, *Black Holes, White Dwarfs and Neutron Stars* shows some plots of Φ_{eff} .

4) You can check Equation 4.5 by taking $\mathbf{\Omega}$ along the \mathbf{z} axis, and considering a particle moving in the $x - y$ plane. Show from Equation 4.4 that $E_J = (v_x'^2 + v_y'^2)/2 + \Phi(\mathbf{x}') - \Omega^2(x'^2 + y'^2)/2$. Write the rate dE_J/dt' at which E_J changes along the particle's path, as measured by the rotating observer: you can use Equations 4.1 and 4.2 to find the

derivatives dx'/dt' , dv'_x/dt' , etc. The rate should be zero, showing that E_J is conserved along the particle's orbit.

4) (Problem 4.4) If the mass \mathcal{M} is replaced by the 'dark halo' potential of Equation 2.20, show that the mass within radius $r \gg a_H$ of its center is $\mathcal{M}(< r) \approx rV_H^2/G$. A satellite with mass $m \ll \mathcal{M}(< D)$ orbits at radius $D \gg a_H$. Substituting the force from the dark halo for that of the point mass \mathcal{M} in Equation 4.7, show that instead of Equation 4.10, we have

$$r_J = D \left[\frac{m}{2\mathcal{M}(< D)} \right]^{1/3}.$$

The Sagittarius dwarf spheroidal galaxy is now about 20 kpc from the Galactic center. Assuming that the rotation curve of the Milky Way remains flat with $V(R) \approx 200 \text{ km s}^{-1}$, show that this dwarf galaxy would need a mass of about $6 \times 10^9 \mathcal{M}_\odot$ if stars 5 kpc from its center are to remain bound to it. Show that this requires $\mathcal{M}/L_V \sim 70$, much larger than the values listed in Table 4.2.

6) Use the virial theorem to show that if a protogalaxy made of stars collapses from rest, the radius of the final galaxy is half of what it was when it began collapsing. By what factor has the density increased? Will the size of the final system be larger or smaller if gas is present that can dissipate energy?

In Q1 of Homework 2, you found that the average density within a sphere at the Sun's orbit was $10^{-5} \rho_{\text{crit}}$. What is the maximum density that the proto-Milky-Way could have had at the moment when it began to collapse?

At the time when collapse began, the density of the protogalaxy had to be greater than that of the universe as a whole (since that was still expanding). What was the average cosmic density at redshift z , given that it is $0.3\rho_{\text{crit}}$ now? Use this to find the earliest redshift at which the Galaxy started to collapse (objects which formed earlier would have been denser than the present Milky Way). According to your calculations above, could the centers of galaxies form earlier than the outer parts, or must they form after the galaxy as a whole has come together?

If the average mass of a galaxy is $10^{11} \mathcal{M}_\odot$, and the density in galaxies is currently $0.01\rho_{\text{crit}}$, what is the average distance between galaxies now?

How does that change with redshift z ?

About how big was the proto-Milky-Way? What is the earliest redshift at which proto-galaxies would be separated from each other and not overlap?