Astro 730 Spring 2006

Homework 2: due 4pm Friday March 3

1) Use Equation 3.20 to find the mass $\mathcal{M}(< R_0)$ within a sphere of radius R_0 about the Galactic center. What is the average density ρ within that sphere, in $\mathcal{M}_\odot\,\mathrm{pc}^{-3}$? Show that this is about 10^5 times larger than the critical density of Equation 1.28. What is the freefall time $1/\sqrt{G\rho}$ of Equation 3.23? (Assembling the material of the inner galaxy must have taken at least this much time.)

Taking $h_R = 4\,\mathrm{kpc}$ in Equation 2.8, show that 60% of the Milky Way's disk lies within the Sun's orbit. Taking $L_V = 5 \times 10^9 L_\odot$ for the bulge and $15 \times 10^9 L_\odot$ for the disk, show that the mass-to-light ratio $\mathcal{M}/L(< R_0) \approx 5$. Using the result of Q4 in Homework 1, explain why we believe that no more than half of the mass within R_0 is dark matter.

The Galaxy's HI disk extends outward to about $2.5R_0$: show that the mass $\mathcal{M}(< 2.5R_0) \approx 2 \times 10^{11}\mathcal{M}_\odot$, so that the mass-to-light ratio $\mathcal{M}/L_V \gtrsim 10$. Physically, why is this $\mathcal{M}/L$ higher than what you found in the first part of the question? What would you expect $\mathcal{M}/L$ to be, if the ratio of luminous to dark matter had its cosmic average value?

2) This question explains how accretion disks work. For a particle in circular orbit in a potential $\Phi(r) = -Kr^{-\alpha}$, where K, α are positive constants, show that $V^2(r) = -\alpha\Phi(r)$. Two gas clouds, masses m_1, m_2, follow circular orbits at radii r_1, r_2, with $r_1 < r_2$. What is the total energy $\mathcal{E}$, and angular momentum $\mathcal{L}$?
The gas clouds are now displaced to different circular orbits at radii $r_1 + \Delta r_1, r_2 + \Delta r_2$. How must $\Delta r_1, \Delta r_2$ be related so that $\mathcal{L}$ is unchanged? Assuming $\Delta r_1, \Delta r_2$ are small, what is the energy change $\Delta\mathcal{E}$?
Show that if $\alpha < 2$, the angular momentum $rV(r)$ of a circular orbit increases with r. We will see in Section 3.3 that this condition is met whenever the circular orbit is stable. In that case, show that the energy of the second state is lower than the initial energy if the inner mass moves inward, so $\Delta r_1 < 0$. Processes that couple different regions of a rotating disk, such as viscosity or spiral structure, can extract energy from the rotation by transferring mass inward and angular momentum outward.

3) If an isolated cluster of stars is initially in equilibrium, and a fraction f of its mass is suddenly removed at each radius, by what factor do the kinetic and potential energies change? If the initial potential energy is $\mathcal{PE}_0$, show that the starting energy $\mathcal{E}_0 = \mathcal{PE}_0/2$, while afterward it is $\mathcal{E}_1 = (1-f)(1-2f)\mathcal{E}_0$. So if $f > 0.5$, the stars are no longer bound together. Use the virial theorem to show that when the remaining stars come to a new equilibrium, the average distance between them is larger by a factor $(1-f)/(1-2f)$.
Complexes of molecular clouds in the Milky Way's disk appear to turn about 10% of the gas into stars. Why are you surprised that *any* bound star clusters exist?

4) Consider the spherical density distribution $\rho_H(r)$ with

$$4\pi G\rho_H(r) = \frac{V_H^2}{r^2 + a_H^2}\,,$$

where V_H and a_H are constants; what is the mass $\mathcal{M}(< r)$ contained within radius r? (Hint: it's OK to guess the form of $V(r)$ and then check your guess.)
Use Equation 3.20 to show that the speed $V(r)$ of a circular orbit at radius r is given by

$$V^2(r) = V_H^2[1 - (a_H/r)\arctan(r/a_H)].$$

This density law is often used to represent the mass of a galaxy's dark halo – why? What major defect does this model have at large radius?

Show that the potential corresponding to the density

$$\rho_{\mathrm{NFW}}(r) = \frac{\rho_N}{(r/a_N)(1+r/a_N)^2} \quad \text{is} \quad \Phi_{\mathrm{NFW}}(r) = -\sigma_N^2 \frac{\ln(1+r/a_N)}{(r/a_N)}\,,$$

where $\sigma_N^2 = 4\pi G\rho_N a_N^2$. How does this model differ from the 'dark halo' near the center $r = 0$? Find the rotation curve $V(r)$: is this or the rotation curve of the 'dark halo' model more like what we measure for the Milky Way in Figure 2.19?