Astro 730 Spring 2006

Homework 1, revised: due in class, Friday February 10

1) Suppose that stars formed within a 100 pc cube follow the Salpeter initial mass function for masses between $\mathcal{M}_l$ and an upper limit $\mathcal{M}_u \gg \mathcal{M}_l$, with no stars of higher or lower masses. Write down and solve the integrals that give (a) the number of stars, (b) their total mass, and (c) the total luminosity, assuming that Equation 1.6 holds with $\alpha \approx 3.5$. Show that the number and mass of stars depend mainly on the mass $\mathcal{M}_l$ of the smallest stars, while the luminosity depends on $\mathcal{M}_u$, the mass of the largest stars. Taking $\mathcal{M}_l = 0.3\mathcal{M}_\odot$ and $\mathcal{M}_u \gg 5\mathcal{M}_\odot$, show that only 2.2% of all stars have $\mathcal{M} > 5\mathcal{M}_\odot$, while these account for 37% of the mass. What fraction of stars have $\mathcal{M} > \mathcal{M}_\odot$?

2) Suppose that stars are born at a constant rate. Assuming $\tau_{\rm gal} = 10\,$Gyr in Equation 2.4 and using Table 1.1 for stellar lifetimes, show that only 11% of all the $2\mathcal{M}_\odot$ stars ever made are still on the main sequence today. What fraction of all the $3\mathcal{M}_\odot$ stars are still there? Now suppose that star formation slows with time t as $e^{-t/t_\star}$, with $t_\star = 3\,$Gyr: show that now only 1.6% of all $2\mathcal{M}_\odot$ stars survive, and merely 0.46% of stars of $3\mathcal{M}_\odot$. For these stars, explain why the initial mass function $\Psi(M_V)$ must be larger for a given observed present-day mass function $\Phi(M_V)$ when the rate of starbirth declines with time, than if it stays constant. How would a gradual slowdown change the inferred $\Psi(M_V)$ for stars longer-lived than the Sun? Explain why $t_\star < 0$ represents an accelerating rate of starbirth. Qualitatively, how would $\Psi(M_V)$ be affected?
$\Phi(M_V)$ is a fairly smooth function, and we have no reason to expect that $\Psi(M_V)$ will have a kink or change in slope near the Sun's luminosity. Together, these imply that star formation locally has not slowed or speeded up by more than a factor of two over the past few gigayears.

3) Suppose that we look at A stars brighter than $m_V = 10$ within 5° of the north Galactic pole. Assuming that they all have $M_V = 0$, to what height $z_{\rm max}$ can we see them? Then $\mathcal{V}_{\rm max} = \Omega z_{\rm max}^3/3$, where $\Omega/4\pi$ is the fraction of the sky covered by our 5° circle. If there are $n(z)$ stars per cubic parsec, show that the number N brighter than $m_V = 10$ is

$$N = \Omega \int_0^{z_{\rm max}} n(z) z^2 dz\,, \text{ while } \left\langle \frac{\mathcal{V}}{\mathcal{V}_{\rm max}} \right\rangle = \frac{1}{N z_{\rm max}^3} \int_0^{z_{\rm max}} n(z) z^2 \cdot \Omega z^3 dz\,.$$

When $n(z)$ is constant, show that $\mathcal{V}/\mathcal{V}_{\rm max} = 0.5$. Suppose that $n(z) = 1$ for $z < 800\,$pc and is zero further away: show that $\mathcal{V}/\mathcal{V}_{\rm max} = 0.26$. What is $\mathcal{V}/\mathcal{V}_{\rm max}$ if $n(z) = 1 - (z/1.5\,\text{kpc})$?

4) Integrating Equation 2.8, show that at radius R the number per unit area (the surface density) of stars of type S is $\Sigma(R, S) = 2n(0, 0, S)h_z(S)\exp[-R/h_R(S)]$. If each has luminosity $L(S)$, the surface brightness $I(R, S) = L(S)\Sigma(R, S)$. Assuming that h_R and h_z are the same for all types of star, show that the disk's total luminosity $L_D = 2\pi I(R = 0)h_{\rm R}^2$. For the Milky Way, taking $L_D = 1.5 \times 10^{10} L_\odot$ in the V band and $h_R = 3\,$kpc, show that the disk's surface brightness at the Sun's position 8 kpc from the center is $\sim 18 L_\odot\,\text{pc}^{-2}$. We will see in Section 3.4 that the mass density in the disk is about $40 - 60\mathcal{M}_\odot\,\text{pc}^{-2}$, so

we have $\mathcal{M}/L_V \sim 2-3$. Why is this larger than $\mathcal{M}/L_V$ for stars within 100 pc of the Sun? (Which stars are found only close to the midplane?)

6) Using an 8-meter telescope to observe the Galactic center regularly over two decades, you notice that one star moves back and forth across the sky in a straight line: its orbit is edge-on. You take spectra to measure its radial velocity V_r, and find that this repeats exactly each time the star is at the same point in the sky. You are in luck: the furthest points of the star's motion on the sky are also when it is closest to the black hole (pericenter) and furthest from it (apocenter), as in the figure (separate file). You measure the separation s of these two points on the sky, and the orbital period P. Assuming that the black hole provides almost all the gravitational force, follow these steps to find both the mass $\mathcal{M}_{\rm BH}$ of the black hole and its distance d from us.
Show that the orbit's semi-major axis $a = 0.5\,{\rm AU} \times (s/1'')(d/1\,{\rm pc})$. You observe $s = 0.248''$: what would a be at a distance of 8 kpc? At the two extremes of its motion across the sky, the star's radial velocity is $V_a = 484\,{\rm km\,s^{-1}}$ and $V_p = 7\,326\,{\rm km\,s^{-1}}$: at which point is it closest to the black hole? The orbit's eccentricity is e; explain why the conservation of angular momentum requires that $V_p(1-e) = V_a(1+e)$, and show that here, $e = 0.876$.
At distance r from the black hole moving at speed V, the star has kinetic energy $\mathcal{KE} = m_\star V^2/2$ and potential energy $\mathcal{PE} = -Gm_\star \mathcal{M}_{\rm BH}/r$. (How do you know it is OK to use the Newtonian formula for this star?) Since the total energy $\mathcal{KE} + \mathcal{PE}$ does not change during the orbit, show that

$$V_p^2 - V_a^2 = \frac{G\mathcal{M}_{\rm BH}}{a} \times \frac{4e}{1-e^2}\,.$$

Measuring v in ${\rm km\,s^{-1}}$, $\mathcal{M}_{\rm BH}$ in $\mathcal{M}_\odot$, and a in parsecs, $G = 4.5 \times 10^{-3}$. Convert a to AU to show that $\mathcal{M}_{\rm BH}/\mathcal{M}_\odot = 3\,822(a/1\,{\rm AU})$. Because $m_\star \ll \mathcal{M}_{\rm BH}$, we can use Kepler's third law: $P^2\,(\text{in years}) = a^3\,(\text{in AU})/\mathcal{M}_{\rm BH}\,(\text{in } \mathcal{M}_\odot)$. You measure $P = 15.24$ yr; use the equation above to eliminate $\mathcal{M}_{\rm BH}/a$ and show that $a = 942$ AU and $\mathcal{M}_{\rm BH} = 3.6 \times 10^6 \mathcal{M}_\odot$. What is the distance to the Galactic center?

5) Here you make a numerical model describing both the distribution of stars and the way we observe them, to explore the *Malmquist bias.* If we observe stars down to a fixed apparent brightness, we do not get a fair mixture of all the stars in the sky, but we include more of the most luminous stars. This method of 'Monte Carlo simulation' is frequently used when a mathematical analysis would be too complex.
(a) Your model sky consists of G-type stars in regions A (70 pc $< d <$ 90 pc), B (90 pc $< d <$ 110 pc), and C (110 pc $< d <$ 130 pc). If the density is uniform, and you have 10 stars in region B, how many are in regions A and C (round to the nearest integer)? For simplicity, let all the stars in region A be at $d = 80$ pc, in B at 100 pc, and in C at 120 pc. G stars do not all have exactly the same luminosity; if the variation corresponds to about 0.3 magnitudes, what fractional change in luminosity is this? For each of your stars, roll a die, note the number N_1 on the upturned face, and give your star $M_V = M_{V,\odot} + 0.2 \times (N_1 - 3.5)$. Find the apparent magnitude of each star. (With a computer you could use more stars, place them randomly in space, and choose the absolute magnitudes from a Gaussian random distribution, with mean $M_{V,\odot}$ and variance 0.3.)

(b) To 'observe' your sky, use a 'telescope' that can 'see' only stars brighter than apparent magnitude $m_V = 9.8$; these stars are your *sample*. How different is their mean absolute magnitude from that for all the stars that you placed in your sky?
What is the average distance of all the stars in your sample? Suppose you assumed that your sample stars each had the average luminosity for all the stars in your sky, and then calculated their distances from their apparent magnitudes: what would you find for their average distance? In which sense would you make an error?

(c) Metal-poor main-sequence stars are bluer for a given luminosity, so they must be fainter at a particular spectral type; if the star's fraction by weight of heavy elements is Z, then $\Delta M_V \approx -0.87 \log_{10}(Z/Z_\odot)$. For each of your stars, roll the die again, note the number N_2, set $Z/Z_\odot = (N_2 + 0.5)/6$, and change the absolute magnitude from part (a) by ΔM_V. Observe the stars again, this time with a telescope that can see to $m_V = 9.7$. For your sample of stars, calculate the average Z of those that fall in region A. Are these on average more or less metal-rich than the stars from region C that are in your sample?
(Errors in measurement have the same kind of effect as a spread in the true luminosity of a class of stars or galaxies. You can make corrections if you know what your measurement errors are; but most people are too optimistic and under-estimate their errors.)