Astronomy 730 / Galaxies

Problem Set 3 Solutions

Problem 1a. S&G 5.2 asks you to convert between mag arcsec\(^{-2}\) and L\(_\odot\) pc\(^{-2}\), argue why surface-brightness is independent of distance, and estimate how much fainter 25 mag arcsec\(^{-2}\) in the I-band is than the brightness of the night sky.

To do the conversion, imagine placing the Sun at 10 pc, and then calculate the number of magnitudes in a hypothetical box 1×1 pc in size. The number of magnitudes is simply the Sun’s absolute magnitude in the I band \(M_V = (V-I)_\odot = 4.11\) mag. The angular size of the 1 pc box is 0.1 radians, so the solid-angle is 0.01 steradians, or \(4.254 \times 10^8\) arcsec\(^2\). Converting to mag arcsec\(^{-2}\) is then \(4.11 + 2.5 \times \log_{10} 4.254 \times 10^8 = 25.68\). Hence 15 mag arcsec\(^{-2}\) is 10.68 mag arcsec\(^{-2}\) brighter, or 18,707 L\(_\odot\) pc\(^{-2}\).

The argument for why this is independent of distance is because if you were to move the Sun to larger distance, its luminosity would decrease in direct proportion to the solid angle subtended by the 1×1 pc box (assuming a Euclidean geometry).

Table 1.9 gives the night sky brightness in the B band to be between 19.4 and 23.4 mag arcsec\(^{-2}\) depending on moon conditions or if from space. This is between 1.6 and 5.6 arcsec\(^{-2}\) brighter than 25 mag arcsec\(^{-2}\), or factors of 4.4 to 173 times brighter in surface-brightness.

Problem 1b. S&G 6.3: From Figure 6.6 its possible to estimate M32 has a central surface-brightness of \(\sim 11.2\) mag arcsec\(^{-2}\) in the V band, while from figure 6.7, the central surface-brightness of NGC 1399 is roughly 16 mag arcsec\(^{-2}\). From S&G 5.2 we have 15.72 mag arcsec\(^{-2}\) in the V band is 18,707 L\(_\odot\) pc\(^{-2}\). From this we find that M32 has a central surface-brightness of roughly \(1.2 \times 10^6\) L\(_\odot\) pc\(^{-2}\) while NGC 1399 has roughly \(14,454\) L\(_\odot\) pc\(^{-2}\).

Problem 2a. S&G 6.6: Equation 3.46 is \(\frac{M}{L} \sim \frac{(9/2\pi)(\sigma_r^2(0)/G I(0) r_c)}{r}\), where we are given \(r_c = 400\) pc, we estimate \(\sigma_r = 360\) km/s from Figure 6.12, and \(I(0) = I_V(0) = 14,454\) L\(_\odot\) pc\(^{-2}\) from the previous problem to get \(M/L\) in the V band. To use \(G = 4.5 \times 10^{-3}\) requires converting km/s to pc/Myr; the conversion factor is of order unity. Plugging in the numbers you find \(M/L_V \sim 7\).

Problem 2b. S&G 6.13: \(D \sim \frac{cz}{H_0} = 3000/75\) Mpc = 40 Mpc. Equation 3.20 is rewritten as \(M(<r) = V^2(r) r/G\), and the text provides \(V(r) = 250\) km/s at \(r = 4'\). At a distance of 40 Mpc, this corresponds to \(r = 46542\) pc. Again converting km/s to pc/Myr yields \(M \sim 7 \times 10^{11}\). Given the B magnitude of 12.02, this corresponds to an absolute B magnitude of \(M_B = 12.02 - 5 \times \log_{10}(40) - 25 = -21.0\), whereas the sun has \(M_B = 4.83 + 0.65 = 5.48\)
mag. The difference in magnitudes is roughly $4 \times 10^{10}$, which is the result leading to $M/L_B \sim 18$.

**Problem 2c. S&G 6.14.** Adopting $\sigma_r \sim 300$ km/s as a reasonable average over the profile in Figure 6.12, we find $V_H = 424$ km/s. Using the mass formula from the previous problem, and again converting to pc/Myr, we find $M \sim 2 \times 10^{12} M_\odot$ within $r = 50$ kpc. We are given that NGC 1399’s $M_V = -21.7$, which we can compare to the Sun’s value of 4.83 mag. The difference is equivalent to about $4.1 \times 10^{10} L_\odot$, which yields $M/L \sim 50$ in the $V$ band.

**Problem 3a.** If $M/L$ is 5 times greater for ellipticals at $z=1$, 8 Gyr in the past, than that implies the luminosity was 5 times higher, assuming the total mass was constant. We are given the numbers that a single burst fades in the B band by about a factor of 3.3 from 0.1 to 1 Gyr, and then another factor of 3 from 1 to 10 Gyr. Since we need a factor of 5 fading over 8 Gyr, then we must be seeing the ellipticals at $z=1$ at an age < 1 Gyr at that redshift.

**b.** Reasonable estimates for the best single-star fits to the spectra in 6.17 and 6.18 are K-giant for the elliptical spectrum, B star for 10 Myr, A star for 100 Myr, F star for 1 Gyr, and somewhere between G and K for 10 Gyr (hard to tell given the scale).

**c.** Figure 6.19 clearly shows that composite stellar types are needed to match both optical and infrared colors. However, given the limited spectral range in Figure 6.17 and 6.18, the one-star mix may not be that bad. For the older stellar populations (and the elliptical), a combination of A+F, F+K, G+K might be a better match. The reasoning is based on what we know about main-sequence lifetimes, and that by 1 Gyr the RGB develops.

**d.** The point here that is relevant for the term project is that you can devise a very simple stellar mix to characterize a simple stellar population of a given age. That means that for the composite stellar population, your book-keeping (the relative numbers of different stars in your model, and the total number of spectral types) can be minimized.

**Problem 4 (a):** Since $\Sigma_{\text{disk}} \propto \frac{\sigma_z^2}{h_z}$ in general for a disk; we assume that today the vertical exponential scale-height, $h_z$ is constant with radius; and $\Sigma_{\text{disk}} = \Sigma = \Sigma_0 \exp(-R/h_R)$, where $\Sigma_0$ is the central mass-surface-density and $h_R$ is the radial disk scale length; it follows that $\sigma_z \propto [h_z \Sigma_0 \exp(-R/h_R)]^{1/2}$. From this it directly follows that $\sigma_z(R,t_0) = \sigma_z(0,t_0) \exp(-R/2h_R)$, or in other words $\sigma_z$ has an exponential radial distribution with twice the scale-length as the mass. $\sigma_z(0,t_0)$ is left to be defined, and will vary for different stellar types. However, $\sigma_z(0,t_0)$ can be determined by taking the value for stars at the solar radius, and knowing the scale-length of the disk. For the Milky Way, if we take the value for old stars in the thin disk at $R_0 \sim 8$ kpc (the solar radius), we have $\sigma_z(R_0,t_0) \sim 20$ km/s. For the Milky Way $h_R \sim 3$ kpc, so $\sigma_z(0,t_0) \sim 76$ km/s (we refine these numbers below).

**b** $\sigma_z(R,t) = \sigma_0 \left(1+t/\tau_H\right)^{1/n}$, where $n$ (the diffusion index) and $\tau_H$ (the heating time-scale) are, in general, functions of radius and time such that $\sigma_z(R,t) = \sigma_z(0,t) \exp(-R/2h_R)$ and
\( \Sigma_{\text{disk}} \propto \sigma_z(R,t)^2 / h_z(R,t) \). At \( t = 0 \) the boundary conditions are \( \sigma_z(R,0) = \sigma_b \) and \( h_z(R,0) = z_b \). At \( t = t_0 \) the boundary conditions are that \( \sigma_z(R,t_0) = \sigma_z(0,t_0) \exp(-R/2h_R) \) and \( h_z(R,t_0) = \text{constant} \). If we take \( \sigma_z(R_0,t_0) = 20 \text{ km/s} \), we know that \( h_z(R,t_0) = 350 \text{ pc} \), again based on the old stars in the thin disk in the solar neighborhood.

(c) Equate \( \sigma_z(R,t_0) = \sigma_z(0,t_0) \exp(-R/2h_R) \) with \( \sigma_z(R,t_0) = \sigma_b \left(1 + t_0/\tau_H\right)^{1/n} \) and solve for \( n \) to find the radial dependence of \( n \) assuming constant \( \tau_H \). This yields \( n = \log_{10}(1 + t_0/\tau_H) / \log_{10} \left[ \sigma_z(0,t_0) \exp(-R/2h_R) / \sigma_b \right] \). This yields a linear relation for \( 1/n \) as a function of \( R \):

\[
1/n = y(R/h_R) = a(R/2h_R) + b, \quad \text{where } a = -0.4343 / \zeta, \quad b = \log_{10} \left[ \sigma_z(0, t_0) / \sigma_b \right] / \zeta, \quad \text{and } \zeta = \log_{10}(1 + t_0/\tau_H).
\]

(d) Following the same equality, but solving for \( \tau_H(R) = t_0 \left\{ \left[ \sigma_z(0, t_0) \exp(-R/2h_R) / \sigma_b \right]^{n-1} \right\}^{1/n} \). In both of the cases here, we have found a solution for \( n \) and \( \tau_H \) that effectively time-averages from \( t = 0 \) to \( t_0 \). If we were to consider \( \sigma_z(R,t) = \sigma_b \left(1 + t/t_\text{H,em} \right)^{1/n} \times \ldots \times \left(1 + t_{m}/t_\text{H,em} \right)^{1/n} \), i.e., a discrete product of \( m \) different time-intervals such that the sum of \( t_1 + \ldots + t_m = t_0 \), we could solve for \( \tau_H(R,t) \) and \( n(R,t) \) for any arbitrary set of time-intervals.

(e) The dynamical time-scale \( t_{\text{dyn}} \) of the disk is comparable to the free-fall time-scale or roughly \( 1/4 \) of the characteristic orbital period. The latter is the easiest to calculate. You may have wondered what is the relevant orbital period, i.e., the radial orbit (\( T_R \approx 2\pi R/V_c \)), where as usual \( V_c \) is the circular speed of the potential at radius \( R \) in the disk), or the vertical orbit (\( T_z \approx h_z/\sigma_z \)). The physical picture is that diffusion arises from the interaction between stars and either molecular clouds, spiral arms, or satellites that changes over time due to differential rotation. In this case, it’s the difference between the orbital periods in radius that are likely to count most. For the Milky Way we can safely assume that \( V_c \approx 220 \text{ km/s} \) outside of the innermost regions of the disk; this is certainly a safe assumption for \( R/h_R = 1 \) and 3. For the MW, setting \( h_R = 3 \text{ kpc} \) then \( t_{\text{dyn}} \approx 0.021 \text{ Gyr at } R/h_R = 1 \), and \( t_{\text{dyn}} \approx 0.063 \text{ Gyr at } R/h_R = 3 \), and we take these as \( \tau_H(R) \) for all time, \( t \).

What we need to do is build a model such that starting with \( \sigma_b \) and \( z_b \) at every radius, the diffusion at each radius is tuned to give the appropriate scale-height, \( h_z \), for a given age of the disk, but that is always constant with radius. Since \( \Sigma_{\text{disk}} \propto \sigma_z^2 / h_z \), alternatively we can couch the required heating in terms of how \( \sigma_z \) changes, which is exactly what we did in (c). We use this relationship for \( n \) with \( \tau_H(R) = t_{\text{dyn}} \), and set \( \sigma_z(0, t_0) \) for the relevant value of \( h_z \) for a given \( t_0 \). Referring to Table 2.1 of S&G, we can take \( h_z \) = 280, 300, 350 pc and \( \sigma_z(R_0, t_0) = 13, 19, 21 \text{ km/s for mean ages of roughly } 3, 6, \text{ and } 10 \text{ Gyr} \). Also following from this table we can adopt \( \sigma_b \) and \( z_b \) to be that of the cold gas, with values of 6 km/s and 65 pc. In our formulation in (c), it turns out we only need to know \( \sigma_b \) because again \( \Sigma_{\text{disk}} \propto \sigma_z^2 / h_z \) and we are assuming \( \Sigma_{\text{disk}} \) is constant with time. The following table steps through the calculation for \( 1/n = y(R/h_R) = a(R/2h_R) + b \):
The larger values of $1/n$ in the previous draft were due to taking $t_{\text{dyn}}$ at $R = 1$ and 3 kpc instead of $R/h_R = 1$ and 3. The values for all $R$ are plotted below. The model gives reasonable values of $1/n$ for $R/h_R < 2$, but at larger radii $1/n$ becomes unreasonably small in the context of a diffusion model.

<table>
<thead>
<tr>
<th>$t_0$ (Gyr)</th>
<th>$h_z$ (pc)</th>
<th>$\sigma_z(R_0, t_0)$ (km/s)</th>
<th>$\sigma_z(0, t_0)$ (km/s)</th>
<th>$\log_{10} [\sigma_z(0, t_0)/\sigma_b]$</th>
<th>$R/h_R=1$</th>
<th>$R/h_R=3$</th>
<th>$R/h_R=1$</th>
<th>$R/h_R=3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>280</td>
<td>49.3</td>
<td>0.91</td>
<td>2.16</td>
<td>1.69</td>
<td>0.32</td>
<td>0.16</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>300</td>
<td>72.1</td>
<td>1.08</td>
<td>2.46</td>
<td>1.98</td>
<td>0.35</td>
<td>0.22</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>350</td>
<td>79.7</td>
<td>1.12</td>
<td>2.68</td>
<td>2.20</td>
<td>0.34</td>
<td>0.21</td>
<td></td>
</tr>
</tbody>
</table>

**Problem 5 (a) S&G 4.6**: Using the virial theorem we have $M(<r) = V^2(r)/2G$, and the problem states $V(r) \sim 200$ km/s and $r = 20$ kpc. This yields $M \sim 1.8 \times 10^{11} M_\odot$. For the stars to be bound to the Sagittarius dSph requires the acceleration from the MW is less than that of the dSph, or $M_{\text{Sag}} = M_{\text{MW}} (r_{\text{Sag}}/r_{\text{MW}})^2$. Since we are given $r_{\text{Sag}} = 5$ kpc, in order to arrive at $M_{\text{Sag}} = 6 \times 10^9 M_\odot$ requires $r_{\text{MW}} = 27.4$ kpc. For the side of the dSph closest to the center of the MW we would have $r_{\text{MW}} = 15$ kpc and $M_{\text{Sag}} = 2 \times 10^{10} M_\odot$. For the side of the dSph farthest to the center of the MW we would have $r_{\text{MW}} = 25$ kpc and $M_{\text{Sag}} = 7.2 \times 10^9 M_\odot$. Both of these yield a more extreme $M/L$ than the value of 70 obtained by taking $L_V = 8 \times 10^7$ from Table 4.1 and $M_{\text{Sag}} = 6 \times 10^9 M_\odot$ (actually, you get $M/L \sim 75$ with S&G’s numbers). The better way to proceed is to use the Jacobi radius (Roche limit) given by equation 4.10 of S&G: $r_J = D [m/(3M+m)]^{1/3}$, using $r_J = 5$ kpc, $D = 20$ kpc, $M = M_{\text{MW}}$ and $m = M_{\text{Sag}}$. (If you used $r_J = D [m/2M]^{1/3}$ that’s fine too.) Solving for $m$ yields a number closer to $6 \times 10^9 M_\odot$. While the numbers are large, Table 4.2 shows there is already a very wide range in $M/L$ for dSph, so contrary to the book, 70 is not out of the question just from the statistics alone.
(b) S&G 4.7: The mean density of the MW at $M_{MW} \approx 10^{11} M_\odot$ and $r_{MW} = 10$ kpc is $2.4 \times 10^{-2} M_\odot / \text{pc}^3$. Since $t_{ff} = \sqrt{1 / G \rho} = 272 \text{ Myr} \sim 300 \text{ Myr}$ for the pre-collapse density which is 8 times higher. Since $t_{ff} \propto \rho^{-1/2} \propto r^{3/2} M^{-1/2}$ we can immediately scale from the MW to Sculptor to find $t_{ff} \sim 300 (2/10)^{3/2} (2 \times 10^7/10^{11})^{-1/2} \text{ Myr} \sim 1.9 \text{ Gyr}$.

(c) S&G 7.1: This problem asks you to calculate the crossing time for galaxies in the Coma cluster. The distance is $2R$, $R = 3$ Mpc, and for half the distance galaxies are typically moving at 800 km/s, and for the other half at 1000 km/s. So $t_{cross} = 2R / \sigma_r \sim 6 \text{ Mpc} / 900 \text{ km/s} \sim 6.7 \text{ Gyr}$, while the Hubble-time $t_H = H_0^{-1} \sim 15 \text{ Gyr}$ for $H_0 = 67$ km/s/Mpc (see S&G eqn. 1.28). Hence $t_H / t_{cross} \sim 2$.

(d) S&G 7.8: From Problem 3.2 we have $r_c \sim 0.644 a_p$. Using $r_c = 200$ kpc from Table 7.1 we have $a_p \sim 310$ kpc. Setting $PE = -2KE$, adopting $PE = -(3\pi/32) GM^2 / a_p$, and $KE = 3/2 M \sigma_r^2$, with $\sigma_r \sim 1000$ km/s, we find $M = (32/\pi G) a_p \sigma_r^2 \sim 7 \times 10^{14} M_\odot$. Since the star-light is given as $500 \times 10^{10} L_\odot$ in the B band, we have $M/L \sim 140 M_\odot / L_\odot$.

(e) S&G 8.1: Equation 1.30 gives $\rho_{crit} = 3 H_0^2 / 8\pi G = 2.8 \times 10^{11} h^2 M_\odot \text{ Mpc}^{-3}$. For the Local Group we have $\rho_{LG} = M_{LG} / V_{LG}$ and $V_{LG} = (4\pi/3) R_{LG}^3$, and we are given $M_{LG} \sim 3 \times 10^{12} M_\odot$ and $R_{LG} = 1$ Mpc. Hence $\rho_{LG} = [3 \times 10^{12} / (4\pi/3)] M_\odot \text{ Mpc}^{-3} \sim 7.2 \times 10^{11} M_\odot \text{ Mpc}^{-3}$. This yields $\rho_{LG} / \rho_{crit} \sim 2.56 h^{-2}$. For $H_0 = 67$ km/s/Mpc, $\rho_{LG} / \rho_{crit} \sim 5.7$, which is more than ample for collapse. Indeed, if you look at Figure 4.2 and the discussion in Section 4.5, it looks like more reasonable values for the Local Group are $M_{LG} \sim 4.5 \times 10^{12} M_\odot$ and $R_{LG} \sim 0.38$ Mpc, yielding $\rho_{LG} / \rho_{crit} \sim 70 h^{-2}$. We can recover the answer in the book if we stick with $R_{LG} = 1$ Mpc and assume that there are 3 sizeable galaxies in the Local Group (MW, M31, M33) each with roughly $2-3 \times 10^{11} M_\odot$ (see Table 5.1, and include each’s retinue of surrounding dwarfs). This ignores the implied dark-matter from the dynamical exercise in Section 4.5. Even if we do ignore the implied dark-matter, a radius of $R_{LG} \sim 0.38$ Mpc still seems more appropriate, yielding $\rho_{LG} / \rho_{crit} \sim 104$ for $H_0 = 67$ km/s/Mpc.

(f) S&G 8.2: The problems gives you that $t_{ff} / t_H \sim 0.03$ for $\rho \sim 10^5 \rho_{crit}$. Since $t_{ff} = \sqrt{1 / G \rho}$ this means that if $\rho_{cluster} \sim 200 \rho_{crit}$ then $t_{ff} / t_H \sim 0.03 (10^5/200)^{1/2} \sim 0.7$, namely there has barely been enough time for the structure to collapse. The argument applies to the Local Group.