Review/Announcements

- Homework #1 handed out today
- Last time
  - Review of various space missions
  - Course goals etc
    - Observing planets
    - Atmospheres
    - Geophysics
What is a planet?

- Nearly spherical – shape determined by self-gravity
- Orbit – low inclination, low eccentricity
- Potential for an atmosphere
- Differentiated interior
- Primary orbit around a star
- Low mass → no fusion
- Clears “zone”
Orbits – a little history

- Ptolemy: Earth-centered with epicycles
- Copernicus: Sun-centered, circular orbits (with a little help from Galileo)
- Kepler: Sun-centered, elliptical orbits
  - Planets orbit in elliptical orbits with Sun at one focus
  - Orbits sweep out equal areas in equal times
  - \( P^2 \) is proportional to \( a^3 \)
- Newton: inverse square law of gravitation
- Einstein: general relativity and the precession of Mercury’s orbit
Newton’s Law $\Rightarrow$ Kepler’s Laws

1. Law of gravitation for $m_1$ and $m_2$...
   1. Derive equation of relative motion
   2. Coordinate change to polar coordinates with an origin on $m_1$
   2. Motion of $m_2$ about $m_1$ lies in a plane perpendicular to the angular momentum vector
   3. Consider $\delta A \sim 1/2r(r+\delta r)\sin \delta \theta \sim r^2/2(\delta \theta)$ (ignoring 2$^{nd}$ and 3$^{rd}$ order terms)
   4. Divide by $\delta t$, and as $\delta t$ goes to 0 we get
      1. $dA/dt = (1/2)r^2(d\theta/dt) = (1/2)h$
      2. $h = \text{constant} \Rightarrow \text{orbits sweep out equal areas in equal times}
Newton’s Law \Rightarrow Kepler’s Laws

- Equation of relative motion in polar coordinates, with \( u = (1/r) \)
  - \( (d^2u/d\theta^2) + u = \mu/h^2 \)
  - \( \mu = G(m_1 + m_2) \)

- Solution is a differential equation with solution:
  - \( u = (\mu/h^2)[1 + e \cos(\theta - \omega)] \)
    - \( e = \) an amplitude
    - \( \omega = \) a phase
Newton’s Law → Kepler’s Laws

- Invert to show that the general solution to an orbit of one mass around another is something that could be an ellipse → Kepler’s first law.

- \( r = \frac{P}{1 + e \cos(\theta - \omega)} \), which is the equation of a conic in polar coordinates
  - Circle: \( e = 0, \ p = a \)
  - Ellipse: \( 0 < e < 1, \ p = a(1 - e^2) \)
  - Parabola: \( e = 1, \ p = 2a \)
  - Hyperbola: \( e > 1, \ p = a(e^2 - 1) \)
Kepler’s Laws

- In general $e << 1$ for “planets”
  - Pluto ($e=0.25$)
  - Mercury ($e=0.21$)
  - Nereid ($e=0.75$)

- $r = \frac{a(1-e^2)}{1-e \cos(\theta-\omega)}$

  - $\theta$ = true longitude = reference direction = “vernal equinox”
  - $\omega$ = longitude of pericenter = angle between periapse and reference direction
  - $f$ = true anomaly = $\theta-\omega$ = angle between object and periapse
  - $a$ = semi-major axis, $b$ = semi-minor axis, $b^2=a^2(1-e^2)$
Example Orbit

Green = Sun

Magenta = pericenter

Semimajor axis = 1.5

Eccentricity = 0.001

True Anomaly = 67 degrees

Argument of pericenter = 43 degrees
Another example

Semimajor axis = 7.3

Eccentricity = 0.23

True anomaly = 12 degrees

Longitude of pericenter = 82

Show that pericenter = 5.621 AU
Kepler’s 3rd Law

- Recall \( \frac{dA}{dt} = (1/2)h \) \( \Rightarrow h^2 = \mu a (1 - e^2) \). If \( A = \pi ab \), then \( T^2 = (4\pi^2/\mu)a^3 \). This is Kepler’s 3rd Law.

- Consider the case of two small objects orbiting a third larger body:
  - \( \frac{(m_2 + m_1)}{(m_3 + m_2)} = \left(\frac{a_1}{a_2}\right)^3 \left(\frac{T_1}{T_2}\right)^2 \)
  - \( m_3 = \text{asteroid}, \ m_2 = \text{asteroid’s moon}, \ m_1 = \text{Galileo probe} \) \( \Rightarrow \) get accurate mass of asteroid \( \Rightarrow \rho \sim 2.6 \text{ g cm}^{-3} \)
More fun with orbits…

- Integrate equation of relative motion…
  
  \[ \frac{1}{2}v^2 - \left( \frac{\mu}{r} \right) = \text{constant} \]

  Just says that orbital energy per mass is conserved

- Define mean motion as \( n = \frac{2\pi}{T} \)

- One can show:
  
  \[ V^2(r) = 2GM\left( \frac{1}{r} - \frac{1}{2a} \right) \quad \text{(vis viva equation)} \]
  
  \[ V_p = na\left[ \frac{(1+e)}{(1-e)} \right]^{1/2} \]
  
  \[ V_a = na\left[ \frac{(1-e)}{(1+e)} \right]^{1/2} \]
More fun with orbits...

- In cartesian coordinates
  - \( x = r \cos f, y = r \sin f \) (\( f \) = true anomaly)
  - \( xdot = -(na/(1-e^2)^{1/2})\sin f \)
  - \( ydot = (na/(1-e^2)^{1/2})(e+\cos f) \)

- Given \( f \) we can calculate the orbital radius and velocity of a body – what we really want is to be able to make predictions about where the body will be in the future.

- Define “eccentric anomaly”, \( E \) = angle between major axis and the radius from the center to the intersection point on a circumscribed circle of radius, \( a \).

- Define “mean anomaly”, \( M = n(t-\tau) \), \( \tau \) is the time of pericenter passage.
  - \( t = \tau, M=f=0 \)
  - \( t = \tau + (T/2), M = f = \pi \)
  - Then \( x = a(\cos E - e), y = a(1-e^2)\sin E \), and \( r=a[1-e\cos E]^{1/2} \)